Discrete mathematics is the branch of mathematics that involves finite or discontinuous quantities. In this unit, you will learn about sequences, series, probability, and statistics.
‘Minesweeper’: Secret to Age-Old Puzzle?

Source: USA TODAY, November 3, 2000

“Minesweeper, a seemingly simple game included on most personal computers, could help mathematicians crack one of the field’s most intriguing problems. The buzz began after Richard Kaye, a mathematics professor at the University of Birmingham in England, started playing Minesweeper. After playing the game steadily for a few weeks, Kaye realized that Minesweeper, if played on a much larger grid, has the same mathematical characteristics as other problems deemed insolvable.” In this project, you will research a mathematician of the past and his or her role in the development of discrete mathematics.

Log on to www.algebra2.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 4.
Sequences and Series

What You’ll Learn

- **Lessons 11-1 through 11-5** Use arithmetic and geometric sequences and series.
- **Lesson 11-6** Use special sequences and iterate functions.
- **Lesson 11-7** Expand powers by using the Binomial Theorem.
- **Lesson 11-8** Prove statements by using mathematical induction.

Why It’s Important

Many number patterns found in nature and used in business can be modeled by sequences, which are lists of numbers. Some sequences are classified by the method used to predict the next term from the previous term(s). When the terms of a sequence are added, a series is formed. In Lesson 11-2, you will learn how the number of seats in the rows of an amphitheater can be modeled using a series.

Key Vocabulary

- arithmetic sequence (p. 578)
- arithmetic series (p. 583)
- sigma notation (p. 585)
- geometric sequence (p. 588)
- geometric series (p. 594)
Prerequisite Skills  To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 11.

For Lessons 11-1 and 11-3  Solve Equations  
(For review, see Lessons 1-3 and 5-5.)

Solve each equation.
1. \(36 = 12 + 4x\)
2. \(-40 = 10 + 5x\)
3. \(12 - 3x = 27\)
4. \(162 = 2x^4\)
5. \(\frac{1}{8} = 4x^5\)
6. \(3x^3 + 4 = -20\)

For Lessons 11-1 and 11-5  Graph Functions  
(For review, see Lesson 2-1.)

Graph each function.
7. \(\{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}\)
8. \(\{(1, -20), (2, -16), (3, -12), (4, -8), (5, -4)\}\)
9. \(\left\{(1, 64), (2, 16), (3, 4), (4, 1), \left(5, \frac{1}{4}\right)\right\}\)
10. \(\left\{(1, 2), (2, 3), \left(3, \frac{7}{2}\right), \left(4, \frac{15}{4}\right), \left(5, \frac{31}{8}\right)\right\}\)

For Lessons 11-1 through 11-5, 11-8  Evaluate Expressions  
(For review, see Lesson 1-1.)

Evaluate each expression for the given value(s) of the variable(s).
11. \(x + (y - 1)z\) if \(x = 3, y = 8,\) and \(z = 2\)
12. \(\frac{x}{2} (y + z)\) if \(x = 10, y = 3,\) and \(z = 25\)
13. \(a \cdot b^{-1}\) if \(a = 2, b = \frac{1}{2},\) and \(c = 7\)
14. \(\frac{a(1 - bc)^2}{1 - b}\) if \(a = -2, b = 3,\) and \(c = 5\)
15. \(\frac{a}{1 - b}\) if \(a = \frac{1}{2},\) and \(b = \frac{1}{6}\)
16. \(\frac{n(n + 1)}{2}\) if \(n = 10\)

Make this Foldable to record information about sequences and series. Begin with one sheet of 11" by 17" paper and four sheets of notebook paper.

**Step 1**  Fold and Cut
Fold the short sides of the 11" by 17" paper to meet in the middle.

**Step 2**  Staple and Label
Fold the notebook paper in half lengthwise. Insert two sheets of notebook paper in each tab and staple edges. Label with lesson numbers.

Reading and Writing  As you read and study the chapter, fill the journal with examples for each lesson.
ARITHMETIC SEQUENCES  The numbers 3, 4, 5, 6, ..., representing the number of shingles in each row, are an example of a sequence of numbers. A sequence is a list of numbers in a particular order. Each number in a sequence is called a term. The first term is symbolized by $a_1$, the second term is symbolized by $a_2$, and so on.

The graph represents the information from the table above. A sequence is a function whose domain is the set of positive integers. You can see from the graph that a sequence is a discrete function.

Many sequences have patterns. For example, in the sequence above for the number of shingles, each term can be found by adding 1 to the previous term. A sequence of this type is called an arithmetic sequence. An arithmetic sequence is a sequence in which each term after the first is found by adding a constant, called the common difference $d$, to the previous term.

Example 1  Find the Next Terms

Find the next four terms of the arithmetic sequence 55, 49, 43, ... .

Find the common difference $d$ by subtracting two consecutive terms.

$49 - 55 = -6$ and $43 - 49 = -6$  So, $d = -6$.

Now add $-6$ to the third term of the sequence, and then continue adding $-6$ until the next four terms are found.

$$
43 + (-6) = 37 \quad 37 + (-6) = 31 \quad 31 + (-6) = 25 \quad 25 + (-6) = 19
$$

The next four terms of the sequence are 37, 31, 25, and 19.

There is a pattern in the way the terms of an arithmetic sequence are formed. It is possible to develop a formula for each term of an arithmetic sequence in terms of the first term $a_1$ and the common difference $d$. Look at the sequence in Example 1.
The following formula generalizes this pattern for any arithmetic sequence.

### Key Concept

**nth Term of an Arithmetic Sequence**

The nth term \( a_n \) of an arithmetic sequence with first term \( a_1 \) and common difference \( d \) is given by

\[
a_n = a_1 + (n - 1)d,
\]

where \( n \) is any positive integer.

### Example 2

**Find a Particular Term**

**CONSTRUCTION** Refer to the information at the left. Assuming that the arithmetic sequence continues, how much would it cost to rent the crane for twelve months?

**Explore** Since the difference between any two successive costs is $15,000, the costs form an arithmetic sequence with common difference 15,000.

**Plan** You can use the formula for the nth term of an arithmetic sequence with \( a_1 = 75,000 \) and \( d = 15,000 \) to find \( a_{12} \), the cost for twelve months.

**Solve**

\[
a_n = a_1 + (n - 1)d \quad \text{Formula for nth term}
\]

\[
a_{12} = 75,000 + (12 - 1)15,000 \quad n = 12, \ a_1 = 75,000, \ d = 15,000
\]

\[
a_{12} = 240,000 \quad \text{Simplify.}
\]

It would cost $240,000 to rent the crane for twelve months.

**Examine** You can find terms of the sequence by adding 15,000. \( a_5 \) through \( a_{12} \) are 135,000, 150,000, 165,000, 180,000, 195,000, 210,000, 225,000, and 240,000. Therefore, $240,000 is correct.

### Example 3

**Write an Equation for the nth Term**

Write an equation for the nth term of the arithmetic sequence 8, 17, 26, 35, … .

In this sequence, \( a_1 = 8 \) and \( d = 9 \). Use the nth term formula to write an equation.

\[
a_n = a_1 + (n - 1)d \quad \text{Formula for nth term}
\]

\[
a_n = 8 + (n - 1)9 \quad a_1 = 8, \ d = 9
\]

\[
a_n = 8 + 9n - 9 \quad \text{Distributive Property}
\]

\[
a_n = 9n - 1 \quad \text{Simplify.}
\]

An equation is \( a_n = 9n - 1 \).
Find Arithmetic Means

Find the four arithmetic means between 16 and 91.

You can use the \( n \)th term formula to find the common difference. In the sequence \( 16, \ldots, 91, \ldots \), \( a_1 \) is 16 and \( a_6 \) is 91.

\[
\begin{align*}
a_n &= a_1 + (n - 1)d \\
a_6 &= 16 + (6 - 1)d \\
91 &= 16 + 5d \\
75 &= 5d \\
15 &= d
\end{align*}
\]

Subtract 16 from each side.

Divide each side by 5.

Now use the value of \( d \) to find the four arithmetic means.

\[
\begin{align*}
16 & \quad 31 \quad 46 \quad 61 \quad 76 \\
+15 & \quad +15 \quad +15 \quad +15
\end{align*}
\]

The arithmetic means are 31, 46, 61, and 76. \( \text{CHECK} \quad 76 + 15 = 91 \)

**Concept Check**

1. Explain why the sequence 4, 5, 7, 10, 14, \ldots is not arithmetic.
2. Find the 15th term in the arithmetic sequence \(-3, 4, 11, 18, \ldots\).
3. \( \text{OPEN ENDED} \) Write an arithmetic sequence with common difference \(-5\).

**Guided Practice**

Find the next four terms of each arithmetic sequence.

4. \(12, 16, 20, \ldots\)
5. \(3, 1, -1, \ldots\)

Find the first five terms of each arithmetic sequence described.

6. \(a_1 = 5, d = 3\)
7. \(a_1 = 14, d = -2\)
8. Find \( a_{13} \) for the arithmetic sequence \(-17, -12, -7, \ldots \).

Find the indicated term of each arithmetic sequence.
9. \( a_1 = 3, d = -5, n = 24 \)
10. \( a_1 = -5, d = 7, n = 13 \)

11. Complete: 68 is the \( \_\_ \)th term of the arithmetic sequence \(-2, 3, 8, \ldots \).

12. Write an equation for the \( n \)th term of the arithmetic sequence \(-26, -15, -4, 7, \ldots \).

13. Find the three arithmetic means between 44 and 92.

14. **ENTERTAINMENT** A basketball team has a halftime promotion where a fan gets to shoot a 3-pointer to try to win a jackpot. The jackpot starts at $5000 for the first game and increases $500 each time there is no winner. Ken has tickets to the fifteenth game of the season. How much will the jackpot be for that game if no one wins by then?

### Practice and Apply

#### Find the next four terms of each arithmetic sequence.
15. \( 9, 16, 23, \ldots \)
16. \( 31, 24, 17, \ldots \)
17. \(-6, -2, 2, \ldots \)
18. \(-8, -5, -2, \ldots \)
19. \(\frac{1}{3}, 1, \frac{5}{3}, \ldots \)
20. \(\frac{18}{5}, \frac{16}{5}, \frac{14}{5}, \ldots \)
21. \(6.7, 6.3, 5.9, \ldots \)
22. \(1.3, 3.8, 6.3, \ldots \)

#### Find the first five terms of each arithmetic sequence described.
23. \( a_1 = 2, d = 13 \)
24. \( a_1 = 41, d = 5 \)
25. \( a_1 = 6, d = -4 \)
26. \( a_1 = 12, d = -3 \)
27. \( a_1 = \frac{4}{3}, d = -\frac{1}{3} \)
28. \( a_1 = \frac{5}{8}, d = \frac{3}{8} \)

29. Find \( a_8 \) if \( a_n = 4 + 3n \).

30. If \( a_n = 1 - 5n \), what is \( a_{10} \)?

#### Find the indicated term of each arithmetic sequence.
31. \( a_1 = 3, d = 7, n = 14 \)
32. \( a_1 = -4, d = -9, n = 20 \)
33. \( a_1 = 35, d = 3, n = 101 \)
34. \( a_1 = 20, d = 4, n = 81 \)
35. \( a_1 = 5, d = \frac{1}{3}, n = 12 \)
36. \( a_1 = \frac{5}{2}, d = -\frac{3}{2}, n = 11 \)
37. \( a_{12} \) for \(-17, -13, -9, \ldots \)
38. \( a_{12} \) for \(8, 3, -2, \ldots \)
39. \( a_{21} \) for \(121, 118, 115, \ldots \)
40. \( a_{43} \) for \(5, 9, 13, 17, \ldots \)

41. **GEOLOGY** Geologists estimate that the continents of Europe and North America are drifting apart at a rate of an average of 12 miles every 1 million years, or about 0.75 inch per year. If the continents continue to drift apart at that rate, how many inches will they drift in 50 years? (\( \text{Hint: } a_1 = 0.75 \))

42. **TOWER OF PISA** To prove that objects of different weights fall at the same rate, Galileo dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. How many feet would an object fall in the tenth second?
Complete the statement for each arithmetic sequence.

43. 170 is the __ ? term of $-4, 2, 8, \ldots$.
44. 124 is the __ ? term of $-2, 5, 12, \ldots$.
45. $-14$ is the __ ? term of $2\frac{1}{5}, 2, 1\frac{4}{5}, \ldots$.

Write an equation for the $n$th term of each arithmetic sequence.

46. $7, 16, 25, 34, \ldots$
47. $18, 11, 4, \ldots$
48. $\frac{3}{5}, \frac{5}{5}, \frac{7}{5}, \ldots$

**GEOMETRY** For Exercises 49–51, refer to the first three arrays of numbers below.

49. Make drawings to find the next three numbers in this pattern.

50. Write an equation representing the $n$th number in this pattern.

51. Is 397 a number in this pattern? Explain.

Find the arithmetic means in each sequence.

52. $55, \ldots, 115$
53. $10, \ldots, -8$
54. $-8, \ldots, 7$
55. $3, \ldots, 27$

56. **CRITICAL THINKING** The numbers $x, y,$ and $z$ are the first three terms of an arithmetic sequence. Express $z$ in terms of $x$ and $y$.

57. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are arithmetic sequences related to roofing?

Include the following in your answer:

- the words that indicate that the numbers of shingles in the rows form an arithmetic sequence, and
- explanations of at least two ways to find the number of shingles in the fifteenth row.

58. What number follows 20 in this arithmetic sequence?

$8, 11, 14, 17, 20, \ldots$

A) 5  B) 23  C) 26  D) 29

59. Find the first term in the arithmetic sequence.

$\ldots, 8\frac{1}{3}, 7, 5\frac{2}{3}, 4\frac{1}{3}, \ldots$

A) 3  B) $9\frac{2}{3}$  C) $10\frac{1}{3}$  D) 11

**Standards Test Practice**

58. What number follows 20 in this arithmetic sequence?

$8, 11, 14, 17, 20, \ldots$

A) 5  B) 23  C) 26  D) 29

59. Find the first term in the arithmetic sequence.

$\ldots, 8\frac{1}{3}, 7, 5\frac{2}{3}, 4\frac{1}{3}, \ldots$

A) 3  B) $9\frac{2}{3}$  C) $10\frac{1}{3}$  D) 11

**Maintain Your Skills**

**Mixed Review**

60. **COMPUTERS** Suppose a computer that costs $3000 new is only worth $600 after 3 years. What is the average annual rate of depreciation? (Lesson 10-6)

Solve each equation. (Lesson 10-5)

61. $3e^x - 2 = 0$
62. $e^{3x} = 4$
63. $\ln (x + 2) = 5$

64. If $y$ varies directly as $x$ and $y = 5$ when $x = 2$, find $y$ when $x = 6$. (Lesson 9-4)

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL** Evaluate each expression for the given values of the variable. (To review evaluating expressions, see Lesson 1-1)

65. $3n - 1; n = 1, 2, 3, 4$
66. $6 - j; j = 1, 2, 3, 4$
67. $4m + 7; m = 1, 2, 3, 4, 5$
11-2 Arithmetic Series

Vocabulary
• series
• arithmetic series
• sigma notation
• index of summation

How do arithmetic series apply to amphitheaters?
The first amphitheaters were built for contests between gladiators. Modern amphitheaters are usually used for the performing arts. Amphitheaters generally get wider as the distance from the stage increases. Suppose a small amphitheater can seat 18 people in the first row and each row can seat 4 more people than the previous row.

ARITHMETIC SERIES  The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is 18 + 22 + 26 + 30 or 96. A series is an indicated sum of the terms of a sequence. Since 18, 22, 26, 30 is an arithmetic sequence, 18 + 22 + 26 + 30 is an arithmetic series. Below are some more arithmetic sequences and the corresponding arithmetic series.

<table>
<thead>
<tr>
<th>Arithmetic Sequence</th>
<th>Arithmetic Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 8, 11, 14, 17</td>
<td>5 + 8 + 11 + 14 + 17</td>
</tr>
<tr>
<td>−9, −3, 3</td>
<td>−9 + (−3) + 3</td>
</tr>
<tr>
<td>3 8 13 18</td>
<td>3 + 8 + 13 + 18</td>
</tr>
<tr>
<td>8 8′ 8′ 8’</td>
<td>8 + 8 + 8 + 8</td>
</tr>
</tbody>
</table>

An arithmetic sequence \( S_n \) has \( n \) terms, and the sum of the first and last terms is \( a_1 + a_n \). Thus, the formula \( S_n = \frac{n}{2}(a_1 + a_n) \) represents the sum of any arithmetic series.

Key Concept  Sum of an Arithmetic Series
The sum \( S_n \) of the first \( n \) terms of an arithmetic series is given by
\[
S_n = \frac{n}{2}[2a_1 + (n - 1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a_1 + a_n).
\]
Example 1  **Find the Sum of an Arithmetic Series**

Find the sum of the first 100 positive integers.

The series is $1 + 2 + 3 + \ldots + 100$. Since you can see that $a_1 = 1$, $a_{100} = 100$, and $d = 1$, you can use either sum formula for this series.

**Method 1**

$$S_n = \frac{n}{2}(a_1 + a_n)$$

**Sum formula**

$$S_{100} = \frac{100}{2}(1 + 100) = 50(101) = 5050$$

The sum of the first 100 positive integers is 5050.

**Method 2**

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

$$S_{100} = \frac{100}{2}[2(1) + (100 - 1)1] = 50(101) = 5050$$

Example 2  **Find the First Term**

**RADIO**  A radio station considered giving away $4000 every day in the month of August for a total of $124,000. Instead, they decided to increase the amount given away every day while still giving away the same total amount. If they want to increase the amount by $100 each day, how much should they give away the first day?

You know the values of $n$, $S_n$, and $d$. Use the sum formula that contains $d$.

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

**Sum formula**

$$S_{31} = \frac{31}{2}[2a_1 + (31 - 1)100]$$

$n = 31, d = 100$

$$124,000 = \frac{31}{2}(2a_1 + 3000)$$

$$8000 = 2a_1 + 3000$$

Multiply each side by $\frac{2}{31}$.

$$5000 = 2a_1$$

Subtract 3000 from each side.

$$a_1 = 2500$$

Divide each side by 2.

The radio station should give away $2500 the first day.

Sometimes it is necessary to use both a sum formula and the formula for the $n$th term to solve a problem.

Example 3  **Find the First Three Terms**

Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$.

**Step 1**  Since you know $a_1, a_n$, and $S_n$, use $S_n = \frac{n}{2}(a_1 + a_n)$ to find $n$.

$S_n = \frac{n}{2}(a_1 + a_n)$

$741 = \frac{n}{2}(9 + 105)$

$741 = 57n$

$n = 13$

**Step 2**  Find $d$.

$a_n = a_1 + (n - 1)d$

$105 = 9 + (13 - 1)d$

$96 = 12d$

$d = 8$

**Step 3**  Use $d$ to determine $a_2$ and $a_3$.

$a_2 = 9 + 8$ or 17

$a_3 = 17 + 8$ or 25

The first three terms are 9, 17, and 25.
**SIGMA NOTATION** Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series 
\[ 3 + 6 + 9 + 12 + \ldots + 30 \]
can be expressed as 
\[ \sum_{n=1}^{10} 3n \]. This expression is read **the sum of 3n as n goes from 1 to 10**.

The variable, in this case \( n \), is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of \( n \) are 1, 2, 3, and so on, through 10.

**Example 4** Evaluate a Sum in Sigma Notation

Evaluate \[ \sum_{j=5}^{8} (3j - 4) \].

**Method 1**
Find the terms by replacing \( j \) with 5, 6, 7, and 8. Then add.
\[
\sum_{j=5}^{8} (3j - 4) = [3(5) - 4] + [3(6) - 4] + [3(7) - 4] + [3(8) - 4]
\]
\[
= 11 + 14 + 17 + 20
\]
\[
= 62
\]

The sum of the series is 62.

**Method 2**
Since the sum is an arithmetic series, use the formula \( S_n = \frac{n}{2}(a_1 + a_n) \).
There are 4 terms, \( a_1 = 3(5) - 4 \) or 11, and \( a_4 = 3(8) - 4 \) or 20.
\[
S_4 = \frac{4}{2}(11 + 20)
\]
\[
S_4 = 62
\]

You can use the sum and sequence features on a graphing calculator to find the sum of a series.

**Graphing Calculator Investigation**

**Sums of Series**
The calculator screen shows the evaluation of \[ \sum_{N=2}^{10} (5N - 2) \]. The first four entries for `seq( are
- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.

The last entry is always 1 for the types of series that we are considering.

**Think and Discuss**
1. Explain why you can use any letter for the index of summation.
2. Evaluate \[ \sum_{n=1}^{8} (2n - 1) \] and \[ \sum_{j=5}^{12} (2j - 9) \]. **Make a conjecture** as to their relationship and explain why you think it is true.
Check for Understanding

Concept Check

1. Explain the difference between a sequence and a series.
2. OPEN ENDED Write an arithmetic series for which \( S_5 = 10 \).
3. OPEN ENDED Write the series \( 7 + 10 + 13 + 16 \) using sigma notation.

Guided Practice

Find \( S_n \) for each arithmetic series described.

4. \( a_1 = 4, a_n = 100, n = 25 \)
5. \( a_1 = 40, n = 20, d = -3 \)
6. \( a_1 = 132, d = -4, a_n = 52 \)
7. \( d = 5, n = 16, a_n = 72 \)

Find the sum of each arithmetic series.

8. \( 5 + 11 + 17 + \ldots + 95 \)
9. \( 38 + 35 + 32 + \ldots + 2 \)
10. \( \sum_{n=1}^{7} (2n + 1) \)
11. \( \sum_{k=3}^{7} (3k + 4) \)

Find the first three terms of each arithmetic series described.

12. \( a_1 = 11, a_n = 110, S_n = 726 \)
13. \( n = 8, a_n = 36, S_n = 120 \)

Application

14. WORLD CULTURES The African-American festival of Kwanzaa includes a ritual involving candles. The first night, a candle is lit and then blown out. The second night, a new candle and the candle from the previous night are lit and blown out. This pattern of lighting a new candle and relighting all the candles from the previous nights is continued for seven nights. Use a formula from this lesson to find the total number of candle lightings during the festival.

Practice and Apply

Homework Help

For Exercises | See Examples
--- | ---
15–32, 39, 40, 45 | 1, 2
33–38 | 4
41–44 | 3

Extra Practice

See page 851.

Find \( S_n \) for each arithmetic series described.

15. \( a_1 = 7, a_n = 79, n = 8 \)
16. \( a_1 = 58, a_n = -7, n = 26 \)
17. \( a_1 = 43, n = 19, a_n = 115 \)
18. \( a_1 = 76, n = 21, a_n = 176 \)
19. \( a_1 = 7, d = -2, n = 9 \)
20. \( a_1 = 3, d = -4, n = 8 \)
21. \( a_1 = 5, d = \frac{1}{2}, n = 13 \)
22. \( a_1 = 12, d = \frac{1}{3}, n = 13 \)
23. \( d = -3, n = 21, a_n = -64 \)
24. \( d = 7, n = 18, a_n = 72 \)
25. \( d = \frac{1}{5}, n = 10, a_n = \frac{23}{10} \)
26. \( d = -\frac{1}{4}, n = 20, a_n = -\frac{53}{12} \)

27. TOYS Jamila is making a triangular wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?

28. CONSTRUCTION A construction company will be fined for each day it is late completing its current project. The daily fine will be \$4000\) for the first day and will increase by \$1000\) each day. Based on its budget, the company can only afford \$60,000\) in total fines. What is the maximum number of days it can be late?

Find the sum of each arithmetic series.

29. \( 6 + 13 + 20 + 27 + \ldots + 97 \)
30. \( 7 + 14 + 21 + 28 + \ldots + 98 \)
31. \( 34 + 30 + 26 + \ldots + 2 \)
32. \( 16 + 10 + 4 + \ldots + (-50) \)
33. \( \sum_{n=1}^{6} (2n + 11) \)
34. \( \sum_{n=1}^{5} (2 - 3n) \)
35. \( \sum_{k=7}^{11} (42 - 9k) \)
36. \( \sum_{i=19}^{25} (5i - 3) \)
37. \( \sum_{i=1}^{300} (7i - 3) \)
38. \( \sum_{k=1}^{150} (11 + 2k) \)
39. Find the sum of the first 1000 positive even integers.
40. What is the sum of the multiples of 3 between 3 and 999, inclusive?

Find the first three terms of each arithmetic series described.
41. \(a_1 = 17, a_n = 197, S_n = 2247\)  
42. \(a_1 = -13, a_n = 427, S_n = 18,423\)  
43. \(n = 31, a_n = 78, S_n = 1023\)  
44. \(n = 19, a_n = 103, S_n = 1102\)

45. **AEROSPACE** On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than in the previous second. How far would an object fall in the first ten seconds after being dropped?

**CRITICAL THINKING** State whether each statement is true or false. Explain.
46. Doubling each term in an arithmetic series will double the sum.
47. Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum.
48. Answer the question that was posed at the beginning of the lesson. How do arithmetic series apply to amphitheaters?
   Include the following in your answer:
   • explanations of what the sequence and the series that can be formed from the given numbers represent, and
   • two ways to find the amphitheater capacity if it has ten rows of seats.

49. \(18, 22, 26, 30, \ldots\)  
50. The angles of a triangle form an arithmetic sequence. If the smallest angle measures 36°, what is the measure of the largest angle?
   \(60°, 72°, 84°, 144°\)

51. Use a graphing calculator to find the sum of each arithmetic series.
   51. \(\sum_{n=21}^{75} (2n + 5)\)  
   52. \(\sum_{n=10}^{50} (3n - 1)\)  
   53. \(\sum_{n=20}^{60} (4n + 3)\)

**Maintain Your Skills**

**Mixed Review** Find the indicated term of each arithmetic sequence. (Lesson 11-1)
54. \(a_1 = 46, d = 5, n = 14\)  
55. \(a_1 = 12, d = -7, n = 22\)

56. **RADIOACTIVITY** The decay of Radon-222 can be modeled by the equation \(y = ae^{-0.1813t}\), where \(t\) is measured in days. What is the half-life of Radon-222? (Lesson 10-6)

57. Solve each equation by completing the square. (Lesson 6-4)
58. \(9x^2 + 96x + 256 = 0\)  
59. \(x^2 - 3x - 20 = 0\)

50. Simplify. (Lesson 5-6)
60. \(\sqrt{3} - 4\sqrt{3}\)  
61. \(\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}\)  
62. \(\sqrt{10} - \sqrt{6} + \sqrt{5} + \sqrt{3}\)

51. **Getting Ready for the Next Lesson** **PREREQUISITE SKILL** Evaluate the expression \(a \cdot b^n - 1\) for the given values of \(a, b,\) and \(n\). (To review evaluating expressions, see Lesson 1-1.)
63. \(a = 1, b = 2, n = 5\)  
64. \(a = 2, b = -3, n = 4\)  
65. \(a = 18, b = \frac{1}{3}, n = 6\)

www.algebra2.com/self_check_quiz
GEOMETRIC SEQUENCES The height of the first rebound of the ball is $3(0.6)$ or 1.8 feet. The height of the second rebound is $1.8(0.6)$ or 1.08 feet. The height of the third rebound is $1.08(0.6)$ or 0.648 feet. The sequence of heights, 1.8, 1.08, 0.648, …, is an example of a geometric sequence. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a constant $r$ called the common ratio.

As with an arithmetic sequence, you can label the terms of a geometric sequence as $a_1, a_2, a_3,$ and so on. The $n$th term is $a_n$ and the previous term is $a_{n-1}$. So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

Example 1 Find the Next Term

Multiple-Choice Test Item

Find the missing term in the geometric sequence: 8, 20, 50, 125, ___.

A. 75  B. 200  C. 250  D. 312.5

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the sequence has a common ratio of 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: $125(2.5) = 312.5$.

The answer is D.

You have seen that each term of a geometric sequence can be expressed in terms of $r$ and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of $r$ and the first term $a_1$. Study the patterns shown in the table on the next page for the sequence 2, 6, 18, 54, ….
You can also use the formula for the \( n \)th term if you know the common ratio and one term of a geometric sequence, but not the first term.

\[ a_n = a_1 \cdot r^{n-1}, \]

where \( n \) is any positive integer.

**Example 2** Find a Particular Term

Find the eighth term of a geometric sequence for which \( a_1 = -3 \) and \( r = -2 \).

\[ a_n = a_1 \cdot r^{n-1} \]

Formula for \( n \)th term

\[ a_8 = (-3) \cdot (-2)^8 - 1 \]

\( n = 8, \ a_1 = -3, \ r = -2 \)

\[ a_8 = (-3) \cdot (-128) \]

\((-2)^7 = -128\)

\[ a_8 = 384 \]

Multiply.

The eighth term is 384.

**Example 3** Write an Equation for the \( n \)th Term

Write an equation for the \( n \)th term of the geometric sequence 3, 12, 48, 192, \ldots .

In this sequence, \( a_1 = 3 \) and \( r = 4 \). Use the \( n \)th term formula to write an equation.

\[ a_n = a_1 \cdot r^{n-1} \]

Formula for \( n \)th term

\[ a_n = 3 \cdot 4^{n-1} \]

\( a_1 = 3, \ r = 4 \)

An equation is \( a_n = 3 \cdot 4^{n-1} \).

You can also use the formula for the \( n \)th term if you know the common ratio and one term of a geometric sequence, but not the first term.

**Example 4** Find a Term Given the Fourth Term and the Ratio

Find the tenth term of a geometric sequence for which \( a_4 = 108 \) and \( r = 3 \).

First, find the value of \( a_1 \).

\[ a_n = a_1 \cdot r^{n-1} \]

Formula for \( n \)th term

\[ a_4 = a_1 \cdot 3^4 - 1 \]

\( n = 4, \ r = 3 \)

\[ 108 = 27a_1 \]

\( a_4 = 108 \)

Divide each side by 27.

Now find \( a_{10} \).

\[ a_n = a_1 \cdot r^{n-1} \]

Formula for \( n \)th term

\[ a_{10} = a_1 \cdot 3^{10} - 1 \]

\( n = 10, \ a_1 = 4, \ r = 3 \)

\[ a_{10} = 78,732 \]

Use a calculator.

The tenth term is 78,732.
GEOMETRIC MEANS  In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called *arithmetic means*. Similarly, the missing term(s) between two nonsuccessive terms of a geometric sequence are called *geometric means*. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, … . You can use the common ratio to find the geometric means in a given sequence.

**Example 5** Find Geometric Means

Find three geometric means between 2.25 and 576.

Use the $n$th term formula to find the value of $r$. In the sequence 2.25, ___, ___, ___, 576, $a_1$ is 2.25 and $a_5$ is 576.

$$a_n = a_1 \cdot r^{n-1} \quad \text{Formula for } n\text{th term}$$

$$a_5 = 2.25 \cdot r^4 \quad n = 5, a_1 = 2.25$$

$$576 = 2.25r^4 \quad a_5 = 576$$

$$256 = r^4 \quad \text{Divide each side by 2.25.}$$

$$\pm 4 = r \quad \text{Take the fourth root of each side.}$$

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of $r$ to find three geometric means.

- $r = 4$
  - $a_2 = 2.25(4)$ or 9
  - $a_3 = 9(4)$ or 36
  - $a_4 = 36(4)$ or 144

- $r = -4$
  - $a_2 = 2.25(-4)$ or $-9$
  - $a_3 = -9(-4)$ or 36
  - $a_4 = 36(-4)$ or $-144$

The geometric means are 9, 36, and 144, or $-9, 36,$ and $-144$.

**Check for Understanding**

**Concept Check**

1. Decide whether each sequence is arithmetic or geometric. Explain.
   a. 1, –2, 4, –8, …
   b. 1, –2, –5, –8, …

2. OPEN ENDED Write a geometric sequence with a common ratio of $\frac{2}{3}$.

3. FIND THE ERROR Marika and Lori are finding the seventh term of the geometric sequence 9, 3, 1, ….

   **Marika**
   
   $$r = \frac{3}{9} \text{ or } \frac{1}{3}$$
   $$a_7 = 9 \left( \frac{1}{3} \right)^{7-1}$$
   $$= \frac{1}{81}$$

   **Lori**
   
   $$r = \frac{3}{2} \text{ or } 3$$
   $$a_7 = 9 \cdot 3^{7-1}$$
   $$= 6561$$

Who is correct? Explain your reasoning.

**Guided Practice** Find the next two terms of each geometric sequence.

4. 20, 30, 45, …
5. $\frac{1}{4}, \frac{1}{2}, -1, …$
6. Find the first five terms of the geometric sequence for which $a_1 = -2$ and $r = 3$.  

---

**Alternate Method**

You may prefer this method. The three means will be $2.25r$, $2.25r^2$, and $2.25r^3$. Then the common ratio is $r = \frac{576}{2.25}$ or $r^4 = \frac{576}{2.25}$. Thus, $r = 4$. 

---

**Study Tip**

**590 Chapter 11 Sequences and Series**
7. Find \(a_9\) for the geometric sequence 60, 30, 15, ….

Find the indicated term of each geometric sequence.

8. \(a_1 = 7, r = 2, n = 4\)
9. \(a_3 = 32, r = -0.5, n = 6\)

10. Write an equation for the \(n\)th term of the geometric sequence 4, 8, 16, ….

11. Find two geometric means between 1 and 27.

12. Find the missing term in the geometric sequence:
   - \(a_1 = 9, a_5 = 99\)

13. Find the first five terms of each geometric sequence described.
   14. \(a_1 = 2, r = -3\)
15. \(a_1 = 405, r = 0.5\)
16. \(a_1 = 1, r = 4\)
17. \(a_1 = 16, a_4 = 256\)
18. \(a_1 = 3, r = 2\)
19. \(a_1 = 576, r = -1/2\)
20. \(a_1 = 10, r = 3\)
21. \(a_1 = 3, r = 5\)
22. \(a_1 = 6, r = 2\)
23. \(a_1 = 5, r = 3\)
24. \(a_1 = 6, r = 4\)
25. \(a_1 = 4, r = 6\)
26. \(a_1 = 5, r = 7\)
27. \(a_1 = 6, r = 8\)
28. \(a_1 = 7, r = 9\)

Find the two geometric means between 1 and 27.

12. Find the missing term in the geometric sequence:
   - \(a_1 = 9, a_5 = 99\)

13. Find the first five terms of each geometric sequence described.
   14. \(a_1 = 2, r = -3\)
15. \(a_1 = 405, r = 0.5\)
16. \(a_1 = 1, r = 4\)
17. \(a_1 = 16, a_4 = 256\)
18. \(a_1 = 3, r = 2\)
19. \(a_1 = 576, r = -1/2\)
20. \(a_1 = 10, r = 3\)
21. \(a_1 = 5, r = 3\)
22. \(a_1 = 6, r = 4\)
23. \(a_1 = 4, r = 5\)
24. \(a_1 = 7, r = 6\)
25. \(a_1 = 5, r = 7\)
26. \(a_1 = 6, r = 8\)
27. \(a_1 = 7, r = 9\)

Find the geometric means in each sequence.

12. Find the missing term in the geometric sequence:
   - \(a_1 = 9, a_5 = 99\)

13. Find the first five terms of each geometric sequence described.
   14. \(a_1 = 2, r = -3\)
15. \(a_1 = 405, r = 0.5\)
16. \(a_1 = 1, r = 4\)
17. \(a_1 = 16, a_4 = 256\)
18. \(a_1 = 3, r = 2\)
19. \(a_1 = 576, r = -1/2\)
20. \(a_1 = 10, r = 3\)
21. \(a_1 = 5, r = 3\)
22. \(a_1 = 6, r = 4\)
23. \(a_1 = 4, r = 5\)
24. \(a_1 = 7, r = 6\)
25. \(a_1 = 5, r = 7\)
26. \(a_1 = 6, r = 8\)
27. \(a_1 = 7, r = 9\)

Find the geometric means in each sequence.

12. Find the missing term in the geometric sequence:
   - \(a_1 = 9, a_5 = 99\)

13. Find the first five terms of each geometric sequence described.
   14. \(a_1 = 2, r = -3\)
15. \(a_1 = 405, r = 0.5\)
16. \(a_1 = 1, r = 4\)
17. \(a_1 = 16, a_4 = 256\)
18. \(a_1 = 3, r = 2\)
19. \(a_1 = 576, r = -1/2\)
20. \(a_1 = 10, r = 3\)
21. \(a_1 = 5, r = 3\)
22. \(a_1 = 6, r = 4\)
23. \(a_1 = 4, r = 5\)
24. \(a_1 = 7, r = 6\)
25. \(a_1 = 5, r = 7\)
26. \(a_1 = 6, r = 8\)
27. \(a_1 = 7, r = 9\)

### More About…

**Art**

The largest ever ice construction was an ice palace built for a carnival in St. Paul, Minnesota, in 1992. It contained 10.8 million pounds of ice.

**Source:** The Guinness Book of Records

**Art**

A one-ton ice sculpture is melting so that it loses one-fifth of its weight per hour. How much of the sculpture will be left after five hours? Write the answer in pounds.

**Salaries**

Geraldo’s current salary is $40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases?

**Write an equation for the \(n\)th term of each geometric sequence.**

35. \(36, 12, 4, \ldots\)
36. \(10, 1, -2, \ldots\)
37. \(4, 6, 9, \ldots\)
38. \(1, 2, 4, \ldots\)
39. \(36, 12, 4, \ldots\)
40. \(16, 4, 1, \ldots\)
41. \(-2, 10, -50, \ldots\)
42. \(4, -12, 36, \ldots\)
43. \(9, \ldots, \ldots, \ldots, 144\)
44. \(4, \ldots, \ldots, \ldots, 324\)
45. \(32, \ldots, \ldots, \ldots, \ldots, 1\)
46. \(3, \ldots, \ldots, \ldots, \ldots, 96\)
**MEDICINE**  For Exercises 47 and 48, use the following information. Iodine-131 is a radioactive element used to study the thyroid gland.

47. **RESEARCH**  Use the Internet or other resource to find the *half-life* of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element.

48. How much of an 80-milligram sample of Iodine-131 would be left after 32 days?

**CRITICAL THINKING**  Determine whether each statement is *true* or *false*. If true, explain. If false, provide a counterexample.

49. Every sequence is either arithmetic or geometric.

50. There is no sequence that is both arithmetic and geometric.

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How do geometric sequences apply to a bouncing ball?
Include the following in your answer:
• the first five terms of the sequence of heights from which the ball falls, and
• any similarities or differences in the sequences for the heights the ball rebounds and the heights from which the ball falls.

52. Find the missing term in the geometric sequence: −5, 10, −20, 40, ___.  
   (A) −80  (B) −35  (C) 80  (D) 100

53. What is the tenth term in the geometric sequence: 144, 72, 36, 18, …?  
   (A) 0  (B) \( \frac{9}{64} \)  (C) \( \frac{9}{32} \)  (D) \( \frac{9}{16} \)

**Maintain Your Skills**

**Mixed Review** Find \( S_n \) for each arithmetic series described.  

54. \( a_1 = 11, a_n = 44, n = 23 \)  
55. \( a_1 = -5, d = 3, n = 14 \)

Find the arithmetic means in each sequence.  

56. 15, ___, ___, 27  
57. −8, ___, ___, ___, −24

58. **GEOMETRY** Find the perimeter of a triangle with vertices at (2, 4), (−1, 3) and (1, −3).  
   (Lesson 8-1)

**Getting Ready for the Next Lesson**  

**PREREQUISITE SKILL** Evaluate each expression.  
   (To review expressions, see Lesson 1-1.)

59. \( \frac{1 - 2^7}{1 - 2} \)  
60. \( \frac{1 - \left(\frac{1}{2}\right)^6}{1 - \left(\frac{1}{2}\right)} \)  
61. \( \frac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)} \)

**Practice Quiz 1**

Find the indicated term of each arithmetic sequence.  

1. \( a_1 = 7, d = 3, n = 14 \)  
2. \( a_1 = 2, d = \frac{1}{2}, n = 8 \)

Find the sum of each arithmetic series described.  

3. \( a_1 = 5, a_n = 29, n = 11 \)  
4. \( 6 + 12 + 18 + \ldots + 96 \)

5. Find \( a_7 \) for the geometric sequence 729, −243, 81, … .  
   (Lesson 11-3)
Limits

You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as \( n \) increases, \( a_n \) approaches 0. The value that the terms of a sequence approach, in this case 0, is called the limit of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

Find the limit of the geometric sequence 1, \( \frac{1}{3} \), \( \frac{1}{9} \), \( \frac{1}{27} \), …

**Step 1  Enter the sequence.**
- The formula for this sequence is \( a_n = \left( \frac{1}{3} \right)^{n-1} \).
- Position the cursor on L1 in the STAT EDIT screen and enter the formula \( \text{seq}(N,N,1,10,1) \). This generates the values 1, 2, …, 10 of the index \( N \).
- Position the cursor on L2 and enter the formula \( \text{seq}((1/3)^{(N-1)},N,1,10,1) \). This generates the first ten terms of the sequence.

**KEYSTROKES:** Review sequences in the Graphing Calculator Investigation on page 585.

Notice that as \( n \) increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for \( n \geq 8 \) the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

**Step 2  Graph the sequence.**
- Use a STAT PLOT to graph the sequence. Use L1 as the Xlist and L2 as the Ylist.

**KEYSTROKES:** Review STAT PLOTS on page 87.

The graph also shows that, as \( n \) increases, the terms approach 0. In fact, for \( n \geq 6 \), the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.

**Exercises**

Use a graphing calculator to find the limit, if it exists, of each sequence.

1. \( a_n = \left( \frac{1}{2} \right)^n \)
2. \( a_n = \left( -\frac{1}{2} \right)^n \)
3. \( a_n = 4^n \)
4. \( a_n = \frac{1}{n^2} \)
5. \( a_n = \frac{2^n}{2^n + 1} \)
6. \( a_n = \frac{n^2}{n + 1} \)

www.algebra2.com/other_calculator_keystrokes
GEOMETRIC SERIES Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is \(1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187\) or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which \(a_1 = 1\) and \(r = 3\). Since 1, 3, 9, 27, 81, 243, 729, 2187 is a geometric sequence, \(1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187\) is called a geometric series. Below are some more examples of geometric sequences and their corresponding geometric series.

<table>
<thead>
<tr>
<th>Geometric Sequences</th>
<th>Geometric Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2, 4, 8, 16</td>
<td>1 + 2 + 4 + 8 + 16</td>
</tr>
<tr>
<td>4, −12, 36</td>
<td>4 + (−12) + 36</td>
</tr>
<tr>
<td>5, 1, (\frac{1}{5}), (\frac{1}{25})</td>
<td>(5 + 1 + \frac{1}{5} + \frac{1}{25})</td>
</tr>
</tbody>
</table>

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above.

\[
S_8 = 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 \\
(−) 3S_8 = 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561 \\
(1 − 3)S_8 = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 − 6561 \\
S_8 = \frac{1 - 6561}{1 - 3} = \frac{-6560}{-2} = 3280
\]

The expression for \(S_8\) can be written as \(S_8 = \frac{a_1 - a_1 r^8}{1 - r}\). A rational expression like this can be used to find the sum of any geometric series.
**Key Concept**

The sum $S_n$ of the first $n$ terms of a geometric series is given by

$$S_n = \frac{a_1 - 1 \cdot r^n}{1 - r} \quad \text{or} \quad S_n = \frac{a_1(1 - r^n)}{1 - r}, \quad \text{where } r \neq 1.$$

You cannot use the formula for the sum with a geometric series for which $r = 1$ because division by 0 would result. In a geometric series with $r = 1$, the terms are constant. For example, $4 + 4 + 4 + \ldots + 4$ is such a series. In general, the sum of $n$ terms of a geometric series with $r = 1$ is $n \cdot a_1$.

**Example 1** Find the Sum of the First $n$ Terms

**GENEALOGY** In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?

Counting your two parents, four grandparents, eight great-grandparents, and so on gives you a geometric series with $a_1 = 2$, $r = 2$, and $n = 15$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{Sum formula}$$

$$S_{15} = \frac{2(1 - 2^{15})}{1 - 2} \quad n = 15, \quad a_1 = 2, \quad r = 2$$

$$S_{15} = 65,534 \quad \text{Use a calculator.}$$

Going back 15 generations, you have 65,534 ancestors.

As with arithmetic series, you can use sigma notation to represent geometric series.

**Example 2** Evaluate a Sum Written in Sigma Notation

Evaluate $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.

**Method 1**

Find the terms by replacing $n$ with 1, 2, 3, 4, 5, and 6. Then add.

$$\sum_{n=1}^{6} 5 \cdot 2^{n-1} = 5(2^{1-1}) + 5(2^{2-1}) + 5(2^{3-1}) + 5(2^{4-1}) + 5(2^{5-1}) + 5(2^{6-1})$$

$$= 5(1) + 5(2) + 5(4) + 5(8) + 5(16) + 5(32)$$

$$= 5 + 10 + 20 + 40 + 80 + 160$$

$$= 315$$

The sum of the series is 315.

**Method 2**

Since the sum is a geometric series, you can use the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

$$S_6 = \frac{5(1 - 2^6)}{1 - 2} \quad n = 6, \quad a_1 = 5, \quad r = 2$$

$$S_6 = \frac{5(-63)}{1 - 1}$$

$$S_6 = 315 \quad \text{Simplify.}$$

How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? Remember the formula for the $n$th term of a geometric sequence or series, $a_n = a_1 \cdot r^{n-1}$. You can use this formula to find an expression involving $r^n$.

$$a_n = a_1 \cdot r^{n-1} \quad \text{Formula for nth term}$$

$$a_n \cdot r = a_1 \cdot r^{n-1} \cdot r \quad \text{Multiply each side by } r.$$
SPECIFIC TERMS  You can use the formula for the sum of a geometric series to help find a particular term of the series.

**Example 3** Use the Alternate Formula for a Sum

Find the sum of a geometric series for which \(a_1 = 15,625\), \(a_n = -5\), and \(r = -\frac{1}{5}\).

Since you do not know the value of \(n\), use the formula derived above.

\[
S_n = \frac{a_1 - a_n r}{1 - r}
\]

Alternate sum formula

\[
= \frac{15,625 - (-5)\left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)} \quad a_1 = 15,625, \quad a_n = -5, \quad r = -\frac{1}{5}
\]

\[
= \frac{15,624}{6} \quad \text{or} \quad 13,020 \quad \text{Simplify.}
\]

**Example 4** Find the First Term of a Series

Find \(a_1\) in a geometric series for which \(S_8 = 39,360\) and \(r = 3\).

\[
S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{Sum formula}
\]

\[
39,360 = \frac{a_1(1 - 3^8)}{1 - 3} \quad S_8 = 39,360; \quad r = 3; \quad n = 8
\]

\[
39,360 = \frac{-6560a_1}{-2} \quad \text{Subtract.}
\]

\[
39,360 = 3280a_1 \quad \text{Divide.}
\]

\[
12 = a_1 \quad \text{Divide each side by 3280.}
\]

The first term of the series is 12.

---

**Check for Understanding**

**Concept Check**

1. **OPEN ENDED** Write a geometric series for which \(r = \frac{1}{2}\) and \(n = 4\).

2. Explain, using geometric series, why the polynomial \(1 + x + x^2 + x^3\) can be written as \(\frac{x^4 - 1}{x - 1}\), assuming \(x \neq 1\).

3. Explain how to write the series \(2 + 12 + 72 + 432 + 2592\) using sigma notation.

**Guided Practice**

Find \(S_n\) for each geometric series described.

4. \(a_1 = 12, \quad a_5 = 972, \quad r = -3\)

5. \(a_1 = 3, \quad a_n = 46,875, \quad r = -5\)

6. \(a_1 = 5, \quad r = 2, \quad n = 14\)

7. \(a_1 = 243, \quad r = -\frac{2}{3}, \quad n = 5\)

Find the sum of each geometric series.

8. \(54 + 36 + 24 + 16 + \ldots\) to 6 terms

9. \(3 - 6 + 12 - \ldots\) to 7 terms

10. \(\sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1}\)

11. \(\sum_{n=1}^{7} \left(\frac{1}{3}\right)^{n-1}\)
Find the indicated term for each geometric series described.
12. \(S_n = \frac{381}{64}, \ r = \frac{1}{2}, \ n = 7; \ a_1\)  
13. \(S_n = 33, \ a_n = 48, \ r = -2; \ a_1\)  
14. **WEATHER** Heavy rain caused a river to rise. The river rose three inches the first day, and each additional day it rose twice as much as the previous day. How much did the river rise in five days?

---

**Practice and Apply**

Find \(S_n\) for each geometric series described.
15. \(a_1 = 2, \ a_6 = 486, \ r = 3\)
16. \(a_1 = 3, \ a_5 = 384, \ r = 2\)
17. \(a_1 = 1296, \ a_n = 1, \ r = -\frac{1}{6}\)
18. \(a_1 = 343, \ a_n = -1, \ r = -\frac{1}{7}\)
19. \(a_1 = 4, \ r = -3, \ n = 5\)
20. \(a_1 = 5, \ r = 3, \ n = 12\)
21. \(a_1 = 2401, \ r = -\frac{1}{7}, \ n = 5\)
22. \(a_1 = 625, \ r = \frac{3}{5}, \ n = 5\)
23. \(a_1 = 162, \ r = \frac{1}{3}, \ n = 6\)
24. \(a_1 = 80, \ r = -\frac{1}{2}, \ n = 7\)
25. \(a_1 = 625, \ r = 0.4, \ n = 8\)
26. \(a_1 = 4, \ r = 0.5, \ n = 8\)
27. \(a_2 = -36, \ a_5 = 972, \ n = 7\)
28. \(a_3 = -36, \ a_6 = -972, \ n = 10\)

29. **HEALTH** Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic and that each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness?

30. **LEGENDS** There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have \$1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth?

Find the sum of each geometric series.
31. \(4096 - 512 + 64 - \ldots\) to 5 terms
32. \(7 + 21 + 63 + \ldots\) to 10 terms
33. \(\frac{1}{16} + \frac{1}{4} + 1 + \ldots\) to 7 terms
34. \(\frac{1}{9} - \frac{1}{3} + 1 - \ldots\) to 6 terms
35. \(\sum_{n=1}^{9} 5 \cdot 2^{n-1}\)
36. \(\sum_{n=1}^{6} 2(-3)^{n-1}\)
37. \(\sum_{n=1}^{7} 144 \left(-\frac{1}{2}\right)^{n-1}\)
38. \(\sum_{n=1}^{8} 64 \left(\frac{3}{4}\right)^{n-1}\)
39. \(\sum_{n=1}^{20} 3 \cdot 2^{n-1}\)
40. \(\sum_{n=1}^{16} 4 \cdot 3^{n-1}\)

Find the indicated term for each geometric series described.
41. \(S_n = 165, \ a_n = 48, \ r = -\frac{2}{3}; \ a_1\)
42. \(S_n = 688, \ a_n = 16, \ r = -\frac{1}{2}; \ a_1\)
43. \(S_n = -364, \ r = -3, \ n = 6; \ a_1\)
44. \(S_n = 1530, \ r = 2, \ n = 8; \ a_1\)
45. \(S_n = 315, \ r = 0.5, \ n = 6; \ a_2\)
46. \(S_n = 249.92, \ r = 0.2, \ n = 5; \ a_3\)

47. **LANDSCAPING** Rob is helping his dad install a fence. He is using a sledgehammer to drive the pointed fence posts into the ground. On his first swing, he drives a post five inches into the ground. Since the soil is denser the deeper he drives, on each swing after the first, he can only drive the post 30% as far into the ground as he did on the previous swing. How far has he driven the post into the ground after five swings?
48. **CRITICAL THINKING** If \( a_1 \) and \( r \) are integers, explain why the value of \( \frac{a_1 - a_1r^n}{1 - r} \) must also be an integer.

49. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

**How is e-mailing a joke like a geometric series?**

Include the following in your answer:
- how the related geometric series would change if each person e-mailed the joke on to four people instead of three, and
- how the situation could be changed to make it better to use a formula than to add terms.

50. **The first term of a geometric series is \(-1\), and the common ratio is \(-3\). How many terms are in the series if its sum is 182?**

   A 6  
   B 7  
   C 8  
   D 9

51. **What is the first term in a geometric series with ten terms, a common ratio of 0.5, and a sum of 511.5?**

   A 64  
   B 128  
   C 256  
   D 512

52. **Use a graphing calculator to find the sum of each geometric series.**

   \[ \sum_{n=1}^{20} 3(-2)^{n-1} \]

   \[ \sum_{n=1}^{15} \left(\frac{1}{2}\right)^{n-1} \]

   \[ \sum_{n=1}^{10} 5(0.2)^{n-1} \]

53. **Find the geometric means in each sequence.** (Lesson 11-3)

   55. \( \frac{1}{24}, \ ? , \ ? , \ ? , \ ? , \ 54 \)

   56. \( -2, \ ? , \ ? , \ ? , \ ? , \ ? , \ ? , \ -\frac{243}{16} \)

54. **Find the sum of each arithmetic series.** (Lesson 11-2)

   57. \( 50 + 44 + 38 + \ldots + 8 \)

   58. \( \sum_{n=1}^{12} (2n + 3) \)

59. **ENTERTAINMENT** For Exercises 59–61, use the table that shows the number of drive-in movie screens in the United States for 1995–2000. (Lesson 2-5)

<table>
<thead>
<tr>
<th>Year</th>
<th>Screens</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>848</td>
</tr>
<tr>
<td>1996</td>
<td>826</td>
</tr>
<tr>
<td>1997</td>
<td>815</td>
</tr>
<tr>
<td>1998</td>
<td>750</td>
</tr>
<tr>
<td>1999</td>
<td>737</td>
</tr>
<tr>
<td>2000</td>
<td>637</td>
</tr>
</tbody>
</table>

   Source: National Association of Theatre Owners

59. Draw a scatter plot, in which \( x \) is the number of years since 1995.

60. Find a prediction equation.

61. Predict the number of screens in 2010.

62. **Online Research Data Update** For the latest statistics on the movie industry, visit: [www.algebra2.com/data_update](http://www.algebra2.com/data_update)

**PREREQUISITE SKILL** Evaluate \( \frac{a}{1 - b} \) for the given values of \( a \) and \( b. \)

(To review evaluating expressions, see Lesson 1-1.)

62. \( a = 1, \ b = \frac{1}{2} \)

63. \( a = 3, \ b = -\frac{1}{2} \)

64. \( a = \frac{1}{3}, \ b = -\frac{1}{3} \)

65. \( a = \frac{1}{2}, \ b = \frac{1}{4} \)

66. \( a = -1, \ b = 0.5 \)

67. \( a = 0.9, \ b = -0.5 \)
Consider the infinite geometric series
\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots
\]
You have already learned how to find the sum \( S_n \) of the first \( n \) terms of a geometric series. For an infinite series, \( S_n \) is called a partial sum of the series. The table and graph show some values of \( S_n \).

### INFINITE GEOMETRIC SERIES

<table>
<thead>
<tr>
<th>( n )</th>
<th>( S_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2} ) or 0.5</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{3}{4} ) or 0.75</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{7}{8} ) or 0.875</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{15}{16} ) or 0.9375</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{31}{32} ) or 0.96875</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{63}{64} ) or 0.984375</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{127}{128} ) or 0.9921875</td>
</tr>
</tbody>
</table>

Notice that as \( n \) increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which \( |r| < 1 \).

**Study Tip**

**Absolute Value**
Recall that \( |r| < 1 \) means \( -1 < r < 1 \).
Let’s look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.

\[ S_n = \frac{a_1 - a_n r^n}{1 - r} \quad \text{Sum of first } n \text{ terms} \]

\[ = \frac{a_1}{1 - r} - \frac{a_n r^n}{1 - r} \quad \text{Write the fraction as a difference of fractions.} \]

If \(-1 < r < 1\), the value of \(r^n\) will approach 0 as \(n\) increases. Therefore, the partial sums of an infinite geometric series will approach \(\frac{a_1}{1 - r} - \frac{a_1(0)}{1 - r}\) or \(\frac{a_1}{1 - r}\). This expression gives the sum of an infinite geometric series.

### Key Concept
**Sum of an Infinite Geometric Series**

The sum \(S\) of an infinite geometric series with \(-1 < r < 1\) is given by

\[ S = \frac{a_1}{1 - r}. \]

An infinite geometric series for which \(|r| \geq 1\) does not have a sum. Consider the series \(1 + 3 + 9 + 27 + 81 + \ldots\). In this series, \(a_1 = 1\) and \(r = 3\). The table shows some of the partial sums of this series. As \(n\) increases, \(S_n\) rapidly increases and has no limit. That is, the partial sums do not approach a particular value.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(S_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>121</td>
</tr>
<tr>
<td>10</td>
<td>29,524</td>
</tr>
<tr>
<td>15</td>
<td>7,174,453</td>
</tr>
<tr>
<td>20</td>
<td>1,743,392,200</td>
</tr>
</tbody>
</table>

### Example 1
**Sum of an Infinite Geometric Series**

Find the sum of each infinite geometric series, if it exists.

a. \(\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \ldots\)

First, find the value of \(r\) to determine if the sum exists.

\(a_1 = \frac{1}{2}\) and \(a_2 = \frac{3}{8}\), so \(r = \frac{3}{4} \quad \text{or} \quad \frac{3}{2}\). Since \(|\frac{3}{4}| < 1\), the sum exists.

Now use the formula for the sum of an infinite geometric series.

\[ S = \frac{\frac{1}{2}}{1 - \frac{3}{4}} = \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{1}{2} \quad \text{Simplify.} \]

The sum of the series is \(\frac{1}{2}\).

b. \(1 - 2 + 4 - 8 + \ldots\)

\(a_1 = 1\) and \(a_2 = -2\), so \(r = \frac{-2}{1} \quad \text{or} \quad -2\). Since \(|-2| \geq 1\), the sum does not exist.
In Lessons 11-2 and 11-4, we used sigma notation to represent finite series. You can also use sigma notation to represent infinite series. An infinity symbol \( \infty \) is placed above the \( \Sigma \) to indicate that a series is infinite.

**Example 2** Infinite Series in Sigma Notation

Evaluate \( \sum_{n=1}^{\infty} 24 \left(-\frac{1}{5}\right)^{n-1} \).

In this infinite geometric series, \( a_1 = 24 \) and \( r = -\frac{1}{5} \).

\[
S = \frac{a_1}{1 - r} \quad \text{Sum formula}
\]
\[
= \frac{24}{1 - \left(-\frac{1}{5}\right)} \quad a_1 = 24, \ r = -\frac{1}{5}
\]
\[
= \frac{24}{6/5} \quad \text{Simplify.}
\]
\[
= 20.
\]

Thus, \( \sum_{n=1}^{\infty} 24 \left(-\frac{1}{5}\right)^{n-1} = 20. \)

**REPEATING DECIMALS** The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction. Remember that decimals with bar notation such as 0.2 and 0.47 represent 0.222222… and 0.474747…, respectively. Each of these expressions can be written as an infinite geometric series.

**Example 3** Write a Repeating Decimal as a Fraction

Write 0.3\(\overline{9} \) as a fraction.

**Method 1**

Write the repeating decimal as a sum.

\[
0.3\overline{9} = 0.393939\ldots
\]
\[
= 0.3 + 0.0039 + 0.000039 + \ldots
\]
\[
= \frac{39}{100} + \frac{39}{10,000} + \frac{39}{1,000,000} + \ldots
\]

In this series, \( a_1 = \frac{39}{100} \) and \( r = \frac{1}{100} \).

\[
S = \frac{a_1}{1 - r} \quad \text{Sum formula}
\]
\[
= \frac{\frac{39}{100}}{1 - \frac{1}{100}} \quad a_1 = \frac{39}{100}, \ r = \frac{1}{100}
\]
\[
= \frac{39}{99} \quad \text{Subtract.}
\]
\[
= \frac{39}{99} \quad \text{Simplify.}
\]

Thus, \( 0.3\overline{9} = \frac{13}{33} \).
Check for Understanding

**Concept Check**

1. **OPEN ENDED** Write the series \(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\) using sigma notation.

2. **Explain** why 0.999999… = 1.

3. **FIND THE ERROR** Miguel and Beth are discussing the series \(-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \ldots\). Miguel says that the sum of the series is \(-\frac{1}{7}\). Beth says that the series does not have a sum. Who is correct? Explain your reasoning.

**Guided Practice**

Find the sum of each infinite geometric series, if it exists.

4. \(a_1 = 36, \ r = \frac{2}{3}\)

5. \(a_1 = 18, \ r = -1.5\)

6. \(16 + 24 + 36 + \ldots\)

7. \(\frac{1}{4} + \frac{1}{6} + \frac{2}{18} + \ldots\)

8. \(6 - 2.4 + 0.96 - \ldots\)

9. \(\sum_{n=1}^{\infty} 40(\frac{3}{5})^{n-1}\)

Write each repeating decimal as a fraction.

10. 0.5

11. 0.73

12. 0.175

**Application**

13. **CLOCKS** Jasmine’s old grandfather clock is broken. When she tries to set the pendulum in motion by holding it against the side of the clock and letting it go, it first swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings?

**Practice and Apply**

**Homework Help**

<table>
<thead>
<tr>
<th>For Exercises</th>
<th>See Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>14–27, 32–39</td>
<td>1</td>
</tr>
<tr>
<td>28–31</td>
<td>2</td>
</tr>
<tr>
<td>40–47</td>
<td>3</td>
</tr>
</tbody>
</table>

**Extra Practice**

See page 852.

Find the sum of each infinite geometric series, if it exists.

14. \(a_1 = 4, \ r = \frac{5}{7}\)

15. \(a_1 = 14, \ r = \frac{7}{3}\)

16. \(a_1 = 12, \ r = -0.6\)

17. \(a_1 = 18, \ r = 0.6\)

18. \(16 + 12 + 9 + \ldots\)

19. \(-8 - 4 - 2 - \ldots\)

20. \(12 - 18 + 24 - \ldots\)

21. \(18 - 12 + 8 - \ldots\)

22. \(1 + \frac{2}{3} + \frac{4}{9} + \ldots\)

23. \(\frac{5}{3} + \frac{25}{3} + \frac{125}{3} + \ldots\)

24. \(\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \ldots\)

25. \(\frac{3}{2} + \frac{3}{4} + \frac{3}{8} - \ldots\)

26. \(3 + 1.8 + 1.08 + \ldots\)

27. \(1 - 0.5 + 0.25 - \ldots\)

28. \(\sum_{n=1}^{\infty} 48\left(\frac{2}{3}\right)^{n-1}\)

29. \(\sum_{n=1}^{\infty} \left(\frac{3}{8}\right)\left(\frac{3}{4}\right)^{n-1}\)

30. \(\sum_{n=1}^{\infty} 3(0.5)^{n-1}\)

31. \(\sum_{n=1}^{\infty} (1.5)(0.25)^{n-1}\)

32. **CHILD’S PLAY** Kimimela’s little sister likes to swing at the playground. Yesterday, Kimimela pulled the swing back and let it go. The swing traveled a distance of 9 feet before heading back the other way. Each swing afterward was only 70% as long as the previous one. Find the total distance the swing traveled.
GEOMETRY  For Exercises 33 and 34, refer to square $ABCD$, which has a perimeter of 40 centimeters. If the midpoints of the sides are connected, a smaller square results. Suppose the process of connecting midpoints of sides and drawing new squares is continued indefinitely.

33. Write an infinite geometric series to represent the sum of the perimeters of all of the squares.

34. Find the sum of the perimeters of all of the squares.

35. AVIATION  A hot-air balloon rises 90 feet in its first minute of flight. In each succeeding minute, it rises only 90% as far as it did during the preceding minute. What is the final height of the balloon?

36. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.

37. The sum of an infinite geometric series is 125, and the value of $r$ is 0.4. Find the first three terms of the series.

38. The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series.

39. The first term of an infinite geometric series is $-8$, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series.

Write each repeating decimal as a fraction.

40. $0.\overline{7}$ 41. $0.\overline{1}$ 42. $0.\overline{36}$ 43. $0.\overline{82}$

44. $0.2\overline{46}$ 45. $0.4\overline{27}$ 46. $0.45$ 47. $0.23\overline{1}$

48. CRITICAL THINKING  Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum $S$ of a general infinite geometric series, multiply each side of the equation by $r$, and subtract equations.

49. WRITING IN MATH  Answer the question that was posed at the beginning of the lesson.

How does an infinite geometric series apply to a bouncing ball?

Include the following in your answer:

• some formulas you might expect to see on the chalkboard if the character in the comic strip really was discussing a bouncing ball, and

• an explanation of how to find the total distance traveled, both up and down, by the bouncing ball described at the beginning of Lesson 11-3.

50. What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$?

   A 3  B 4  C 9  D 12

51. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots =$

   A $\frac{3}{2}$  B $\frac{80}{27}$  C 3  D does not exist

www.algebra2.com/self_check_quiz
Maintain Your Skills

Find $S_n$ for each geometric series described.  \textit{(Lesson 11-4)}

52. $a_1 = 1, a_6 = -243, r = -3$  
53. $a_1 = 72, r = \frac{1}{3}, n = 7$

54. **PHYSICS** A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes of the piston?  \textit{(Lesson 11-3)}

Solve each equation or inequality. Check your solution.  \textit{(Lesson 10-1)}

55. $6^x = 216$  
56. $2^x = \frac{1}{8}$  
57. $3^x - 2 \geq 27$

Simplify each expression.  \textit{(Lesson 9-2)}

58. $\frac{-2}{ab} + \frac{5}{a^2}$  
59. $\frac{1}{x - 3} - \frac{2}{x + 1}$  
60. $\frac{1}{x^2 + 6x + 8} + \frac{3}{x + 4}$

Write an equation for the circle that satisfies each set of conditions.  \textit{(Lesson 8-3)}

61. center (2, 4), radius 6  
62. endpoints of a diameter at (7, 3) and (-1, -5)

Find all the zeros of each function.  \textit{(Lesson 7-5)}

63. $f(x) = 8x^3 - 36x^2 + 22x + 21$  
64. $g(x) = 12x^4 + 4x^3 - 3x^2 - x$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are integers.  \textit{(Lesson 6-3)}

65. 6, -6  
66. -2, -7  
67. 6, 4

**RECREATION** For Exercises 68 and 69, refer to the graph at the right.  \textit{(Lesson 2-3)}

68. Find the average rate of change of the number of visitors to Yosemite National Park from 1996 to 1999.
69. Was the number of visitors increasing or decreasing from 1996 to 1999?

**USA TODAY Snapshots®**

**Yosemite visitors peak in '96**

Visitors at Yosemite National Park:

<table>
<thead>
<tr>
<th>Year</th>
<th>Visitors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1993</td>
<td>3,983,749</td>
</tr>
<tr>
<td>1994</td>
<td>4,105,755</td>
</tr>
<tr>
<td>1995</td>
<td>4,101,928</td>
</tr>
<tr>
<td>1996</td>
<td>4,190,557</td>
</tr>
<tr>
<td>1997</td>
<td>3,801,397</td>
</tr>
<tr>
<td>1998</td>
<td>3,792,754</td>
</tr>
<tr>
<td>1999</td>
<td>3,648,384</td>
</tr>
</tbody>
</table>

Source: Yosemite National Park  By Hilary Wasson and Quin Tian, USA TODAY

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL** Find each function value.  \textit{(To review evaluating functions, see Lesson 2-1.)}

70. $f(x) = 2x, f(1)$  
71. $g(x) = 3x - 3, g(2)$  
72. $h(x) = -2x + 2, h(0)$  
73. $f(x) = 3x - 1, f\left(\frac{1}{2}\right)$  
74. $g(x) = x^2, g(2)$  
75. $h(x) = 2x^2 - 4, h(0)$
Amortizing Loans

When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the principal, or original amount of the loan. This process is called amortization. You can use a spreadsheet to analyze the payments, interest, and balance on a loan. A table that shows this kind of information is called an amortization schedule.

Example

Marisela just bought a new sofa for $495. The store is letting her make monthly payments of $43.29 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be \( \frac{9\%}{12} \) or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by \( 1 + 0.0075 \) or 1.0075 and then subtracting 43.29.

In a spreadsheet, use the column of numbers for the number of payments and use column B for the balance. Enter the interest rate and monthly payment in cells in column A so that they can be easily updated if the information changes.

The spreadsheet at the right shows the formulas for the balances after each of the first six payments. After six months, Marisela still owes $253.04.

Exercises

1. Let \( b_n \) be the balance left on Marisela’s loan after \( n \) months. Write an equation relating \( b_n \) and \( b_{n+1} \).

2. Payments at the beginning of a loan go more toward interest than payments at the end. What percent of Marisela’s loan remains to be paid after half a year?

3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?

4. Suppose Marisela decides to pay $50 every month. How long would it take her to pay off the loan?

5. Suppose that, based on how much she can afford, Marisela will pay a variable amount each month in addition to the $43.29. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.

6. Jamie has a three-year, $12,000 car loan. The annual interest rate is 6%, and his monthly payment is $365.06. After twelve months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?
SPECIAL SEQUENCES  Notice that the sequence 1, 1, 2, 3, 5, 8, 13, … has a pattern. Each term in the sequence is the sum of the two previous terms. For example, 8 = 3 + 5 and 13 = 5 + 8. This sequence is called the Fibonacci sequence, and it is found in many places in nature.

The first five terms of the sequence are 4, 10, 28, 82, and 244. The formula \(a_n = a_{n-2} + a_{n-1}\) is an example of a recursive formula. This means that each term is formulated from one or more previous terms. To be able to use a recursive formula, you must be given the value(s) of the first term(s) so that you can start the sequence and then use the formula to generate the rest of the terms.

**Example 1**  Use a Recursive Formula

Find the first five terms of the sequence in which \(a_1 = 4\) and \(a_{n+1} = 3a_n - 2\), \(n \geq 1\).

\[
a_{n+1} = 3a_n - 2  \quad \text{Recursive formula}
\]

\[
a_1 + 1 = 3a_1 - 2 \quad n = 1 \\
a_2 = 3(4) - 2 \quad a_1 = 4
\]

\[
a_2 + 1 = 3a_2 - 2 \quad n = 2 \\
a_3 = 3(10) - 2 \quad a_2 = 10
\]

\[
a_3 + 1 = 3a_3 - 2 \quad n = 3 \\
a_4 = 3(28) - 2 \quad a_3 = 28
\]

\[
a_4 + 1 = 3a_4 - 2 \quad n = 4 \\
a_5 = 3(82) - 2 \quad a_4 = 82
\]

The first five terms of the sequence are 4, 10, 28, 82, and 244.
**Example 2**  
Find and Use a Recursive Formula

**GARDENING** Mr. Yazaki discovered that there were 225 dandelions in his garden on the first Saturday of spring. He had time to pull out 100, but by the next Saturday, there were twice as many as he had left. Each Saturday in spring, he removed 100 dandelions, only to find that the number of remaining dandelions had doubled by the following Saturday.

a. Write a recursive formula for the number of dandelions Mr. Yazaki finds in his garden each Saturday.

Let \( d_n \) represent the number of dandelions at the beginning of the \( n \)th Saturday. Mr. Yazaki will pull 100 of these out of his garden, leaving \( d_n - 100 \). The number \( d_{n+1} \) of dandelions the next Saturday will be twice this number. So, 
\[
d_{n+1} = 2(d_n - 100) \text{ or } 2d_n - 200.
\]

b. Find the number of dandelions Mr. Yazaki would find on the fifth Saturday.

On the first Saturday, there were 225 dandelions, so \( d_1 = 225 \).

\[
d_{n+1} = 2d_n - 200 \quad \text{Recursive formula}
\]

\[
d_{1+1} = 2d_1 - 200 \quad n = 1 \quad d_{3+1} = 2d_3 - 200 \quad n = 3
\]
\[
d_2 = 2(225) = 200 \text{ or } 250 \quad d_4 = 2(300) - 200 \text{ or } 400
\]
\[
d_{2+1} = 2d_2 - 200 \quad n = 2 \quad d_{4+1} = 2d_4 - 200 \quad n = 4
\]
\[
d_3 = 2(250) = 200 \text{ or } 300 \quad d_5 = 2(400) - 200 \text{ or } 600
\]

On the fifth Saturday, there would be 600 dandelions in Mr. Yazaki’s garden.

You can use sequences to analyze some games.

### Algebra Activity

**Special Sequences**

The object of the *Towers of Hanoi* game is to move a stack of \( n \) coins from one position to another in the fewest number of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may not be placed on top of a dime.

**Model and Analyze**

1. Draw three circles on a sheet of paper, as shown above. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle?

2. Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.)

3. Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle?

**Make a Conjecture**

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number \( a_n \) of moves required to move a stack of \( n \) coins.
**ITERATION**  
**Iteration** is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is \( f \circ f(x) \) or \( f(f(x)) \). If you compose a function with itself two times, the result is \( f \circ f \circ f(x) \) or \( f(f(f(x))) \), and so on.

You can use iteration to recursively generate a sequence. Start with an initial value \( x_0 \). Let \( x_1 = f(x_0) \), \( x_2 = f(x_1) \) or \( f(f(x_0)) \), \( x_3 = f(x_2) \) or \( f(f(f(x_0))) \), and so on.

**Example 3  Iterate a Function**

Find the first three iterates \( x_1, x_2, x_3 \) of the function \( f(x) = 2x + 3 \) for an initial value of \( x_0 = 1 \).

To find the first iterate \( x_1 \), find the value of the function for \( x_0 = 1 \).

\[
x_1 = f(x_0) \\
= f(1) \\
= 2(1) + 3 \quad \text{Simplify.}
\]

To find the second iterate \( x_2 \), substitute \( x_1 \) for \( x \).

\[
x_2 = f(x_1) \\
= f(5) \\
= 2(5) + 3 \quad \text{Simplify.}
\]

Substitute \( x_2 \) for \( x \) to find the third iterate.

\[
x_3 = f(x_2) \\
= f(13) \\
= 2(13) + 3 \quad \text{Simplify.}
\]

Therefore, 1, 5, 13, 29 is an example of a sequence generated using iteration.

---

**Check for Understanding**

**Concept Check**

1. Write recursive formulas for the \( n \)th terms of arithmetic and geometric sequences.

2. **OPEN ENDED** Write a recursive formula for a sequence whose first three terms are 1, 1, and 3.

3. State whether the statement \( x_n \neq x_{n-1} \) is sometimes, always, or never true if \( x_n = f(x_{n-1}) \). Explain.

**Guided Practice**

Find the first five terms of each sequence.

4. \( a_1 = 12, a_{n+1} = a_n - 3 \)
5. \( a_1 = -3, a_{n+1} = a_n + n \)
6. \( a_1 = 0, a_{n+1} = -2a_n - 4 \)
7. \( a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3a_n \)

Find the first three iterates of each function for the given initial value.

8. \( f(x) = 3x - 4, x_0 = 3 \)
9. \( f(x) = -2x + 5, x_0 = 2 \)
10. \( f(x) = x^2 + 2, x_0 = -1 \)

**Application**  
**BANKING** For Exercises 11 and 12, use the following information.

Rita has deposited $1000 in a bank account. At the end of each year, the bank posts interest to her account in the amount of 5% of the balance, but then takes out a $10 annual fee.

11. Let \( b_0 \) be the amount Rita deposited. Write a recursive equation for the balance \( b_n \) in her account at the end of \( n \) years.

12. Find the balance in the account after four years.
Find the first five terms of each sequence.

13. \(a_1 = -6\), \(a_{n+1} = a_n + 3\)
14. \(a_1 = 13\), \(a_{n+1} = a_n + 5\)
15. \(a_1 = 2\), \(a_{n+1} = a_n - n\)
16. \(a_1 = 6\), \(a_{n+1} = a_n + n + 3\)
17. \(a_1 = 9\), \(a_{n+1} = 2a_n - 4\)
18. \(a_1 = 4\), \(a_{n+1} = 3a_n - 6\)
19. \(a_1 = -1\), \(a_2 = 5\), \(a_{n+1} = a_n + a_{n-1}\)
20. \(a_1 = 4\), \(a_2 = -3\), \(a_{n+2} = a_{n+1} + 2a_n\)
21. \(a_1 = \frac{7}{2}\), \(a_{n+1} = -\frac{n}{n+1} \cdot a_n\)
22. \(a_1 = \frac{3}{4}\), \(a_{n+1} = \frac{n^2 + 1}{n} \cdot a_n\)
23. If \(a_0 = 7\) and \(a_{n+1} = a_n + 12\) for \(n \geq 0\), find the value of \(a_5\).
24. If \(a_0 = 1\) and \(a_{n+1} = -2.1\) for \(n \geq 0\), then what is the value of \(a_4\)?

GEOMETRY For Exercises 25 and 26, use the following information.
Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process of drawing a square along a long side of the rectangle formed at the previous step.

25. Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with the lengths of the sides of the two original squares.
26. Identify the sequence in Exercise 25.

27. LOANS The Cruz family is taking out a mortgage loan for $100,000 to buy a house. Their monthly payment is $678.79. The recursive formula \(b_n = 1.006b_{n-1} - 678.79\) describes the balance left on the loan after \(n\) payments. Find the balances of the loan after each of the first eight payments.

GEOMETRY For Exercises 28–30, study the triangular numbers shown below.

28. Write a sequence of the first five triangular numbers.
29. Write a recursive formula for the \(n\)th triangular number \(t_n\).
30. What is the 200th triangular number?

Real Estate Agent
Most real estate agents are independent business-people who earn their income from commission.

Online Research
To learn more about a career in real estate, visit: www.algebra2.com/careers
Find the first three iterates of each function for the given initial value.

31. \( f(x) = 9x - 2, \ x_0 = 2 \)  
32. \( f(x) = 4x - 3, \ x_0 = 2 \)
33. \( f(x) = 3x + 5, \ x_0 = -4 \)  
34. \( f(x) = 5x + 1, \ x_0 = -1 \)
35. \( f(x) = 2x^2 - 5, \ x_0 = -1 \)  
36. \( f(x) = 3x^2 - 4, \ x_0 = 1 \)
37. \( f(x) = 2x^2 + 2x + 1, \ x_0 = \frac{1}{2} \)  
38. \( f(x) = 3x^2 - 3x + 2, \ x_0 = \frac{1}{3} \)

39. **ECONOMICS** If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function \( c(x) = 1.02x \). Find the cost of a $70 portable stereo in four years if the rate of inflation remains constant.

40. **CRITICAL THINKING** Are there a function \( f(x) \) and an initial value \( x_0 \) such that the first three iterates, in order, are 4, 4, and 7? If so, state such a function and initial value. If not, explain.

41. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

**How is the Fibonacci sequence illustrated in nature?**

Include the following in your answer:

- the 13th term in the Fibonacci sequence, with an explanation of what it tells you about the plant described, and
- an explanation of why the Fibonacci sequence is neither arithmetic nor geometric.

---

**Standardized Test Practice**

42. If \( a \) is positive, what percent of \( 4a \) is 8?

- **A** \( \frac{a}{100} \% \)
- **B** \( \frac{a}{2} \% \)
- **C** \( 8\% \)
- **D** \( \frac{200}{a} \% \)

43. The figure at the right is made of three concentric semicircles. What is the total area of the shaded regions?

- **A** \( 4\pi \) units\(^2\)
- **B** \( 10\pi \) units\(^2\)
- **C** \( 12\pi \) units\(^2\)
- **D** \( 20\pi \) units\(^2\)

---

**Maintain Your Skills**

**Mixed Review** Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

44. \( 9 + 6 + 4 + \ldots \)
45. \( \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \ldots \)
46. \( 4 - \frac{8}{3} + \frac{16}{9} + \ldots \)

Find the sum of each geometric series. (Lesson 11-4)

47. \( 2 - 10 + 50 - \ldots \) to 6 terms
48. \( 3 + \frac{1}{3} + \ldots \) to 7 terms

49. **GEOMETRY** The area of rectangle \( ABCD \) is \( 6x^2 + 38x + 56 \) square units. Its width is \( 2x + 8 \) units. What is the length of the rectangle? (Lesson 5-3)

---

**Getting Ready for the Next Lesson** **BASIC SKILL** Evaluate each expression.

50. \( 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \)
51. \( 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \)
52. \( \frac{4 \cdot 3}{2 \cdot 1} \)
53. \( \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \)
54. \( \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} \)
55. \( \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \)
Fractals

Fractals are sets of points that often involve intricate geometric shapes. Many fractals have the property that when small parts are magnified, the detail of the fractal is not lost. In other words, the magnified part is made up of smaller copies of itself. Such fractals can be constructed recursively.

You can use isometric dot paper to draw stages of the construction of a fractal called the von Koch snowflake.

**Stage 1** Draw an equilateral triangle with sides of length 9 units on the dot paper.

**Stage 2** Now remove the middle third of each side of the triangle from Stage 1 and draw the other two sides of an equilateral triangle pointing outward.

Imagine continuing this process indefinitely. The von Koch snowflake is the shape that these stages approach.

**Model and Analyze**

1. Copy and complete the table. Draw Stage 3, if necessary.

<table>
<thead>
<tr>
<th>Stage</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Segments</td>
<td>3</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of each Segment</td>
<td>9</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perimeter</td>
<td>27</td>
<td>36</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Write recursive formulas for the number $s_n$ of segments in Stage $n$, the length $\ell_n$ of each segment in Stage $n$, and the perimeter $P_n$ of Stage $n$.

3. Write nonrecursive formulas for $s_n$, $\ell_n$, and $P_n$.

4. What is the perimeter of the von Koch snowflake? Explain.

5. Explain why the area of the von Koch snowflake can be represented by the infinite series $\frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + 3\sqrt{3} + \frac{4\sqrt{3}}{3} + \ldots$.

6. Find the sum of the series in Exercise 5. Explain your steps.

7. Do you think the results of Exercises 4 and 6 are contradictory? Explain.
Pascal's Triangle

Although he did not discover it, Pascal's triangle is named for the French mathematician Blaise Pascal (1623–1662).

Pascal's Triangle
You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Remember that a binomial is a polynomial with two terms. Expand a few powers of the binomial \((b + g)^n\).

\[
\begin{align*}
(b + g)^0 &= 1b^0g^0 \\
(b + g)^1 &= b^1g^0 + 1b^0g^1 \\
(b + g)^2 &= b^2g^0 + 2b^1g^1 + 1b^0g^2 \\
(b + g)^3 &= b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3 \\
(b + g)^4 &= b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4
\end{align*}
\]

The coefficient 6 of the \(b^2g^2\) term in the expansion of \((b + g)^4\) gives the number of sequences of births that result in two boys and two girls. As another example, the coefficient 4 of the \(b^1g^3\) term gives the number of sequences with one boy and 3 girls.

Here are some patterns that can be seen in any binomial expansion of the form \((a + b)^n\).

1. There are \(n + 1\) terms.
2. The exponent \(n\) of \((a + b)^n\) is the exponent of \(a\) in the first term and the exponent of \(b\) in the last term.
3. In successive terms, the exponent of \(a\) decreases by one, and the exponent of \(b\) increases by one.
4. The sum of the exponents in each term is \(n\).
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

The coefficients form a pattern that is often displayed in a triangular formation. This is known as Pascal's triangle. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.

\[
\begin{array}{ccccccc}
(a + b)^0 & 1 \\
(a + b)^1 & 1 & 1 \\
(a + b)^2 & 1 & 2 & 1 \\
(a + b)^3 & 1 & 3 & 3 & 1 \\
(a + b)^4 & 1 & 4 & 6 & 4 & 1 \\
(a + b)^5 & 1 & 5 & 10 & 10 & 5 & 1
\end{array}
\]
**Example 1** Use Pascal’s Triangle

Expand \((x + y)^7\).

Write two more rows of Pascal’s triangle.

\[
\begin{array}{ccccccc}
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
\]

Use the patterns of a binomial expansion and the coefficients to write the expansion of \((x + y)^7\).

\[
(x + y)^7 = 1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7
\]

---

**THE BINOMIAL THEOREM** Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

\[
\begin{array}{ccccccc}
(a + b)^0 & 1 & & & & & \\
(a + b)^1 & 1 & 1 & & & & \\
(a + b)^2 & 1 & 2 & 1 & & & \\
(a + b)^3 & 1 & 3 & 3 & 1 & & \\
(a + b)^4 & 1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]

This pattern provides the coefficients of \((a + b)^n\) for any nonnegative integer \(n\). The pattern is summarized in the **Binomial Theorem**.

**Key Concept**

If \(n\) is a nonnegative integer, then

\[
(a + b)^n = 1a^n b^0 + \frac{n}{1 \cdot 2} a^{n-1} b^1 + \frac{n(n-1)}{1 \cdot 2 \cdot 3} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-3} b^3 + \ldots + 1a^0 b^n.
\]

---

**Example 2** Use the Binomial Theorem

Expand \((a - b)^6\).

The expansion will have seven terms. Use the sequence \(1, 6, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}\) to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

\[
(a - b)^6 = 1a^6(-b)^0 + \frac{6 \cdot 5}{1 \cdot 2} a^5(-b)1 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} a^4(-b)^2 + \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} a^3(-b)^3 + \ldots + 1a^0(-b)^6
\]

\[
= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6
\]

Notice that in terms having the same coefficients, the exponents are reversed, as in \(15a^2b^4\) and \(15a^2b^4\).

---

The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product \(4 \cdot 3 \cdot 2 \cdot 1\) is written \(4!\) and is read \(4\) factorial. In general, if \(n\) is a positive integer, then \(n! = n(n - 1)(n - 2)(n - 3) \ldots 2 \cdot 1\).

By definition, \(0! = 1\).
**Example 3** Factorials

Evaluate \( \frac{8!}{351!} \).

\[
\frac{8!}{351!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56
\]

Note that \( 8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \), so \( \frac{8!}{351!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} \) or \( 8 \cdot 7 \cdot 6 

An expression such as \( \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \) in Example 2 can be written as a quotient of factorials. In this case, \( \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!} \). Using this idea, you can rewrite the expansion of \((a + b)^n\) using factorials.

\[
(a + b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k}b^k
\]

You can also write this series using sigma notation.

In general, the Binomial Theorem can be written both in factorial notation and in sigma notation.

**Key Concept** Binomial Theorem, Factorial Form

\[
(a + b)^n = \frac{n!}{n!} a^n b^0 + \frac{n!}{(n-1)!1!} a^{n-1} b^1 + \frac{n!}{(n-2)!2!} a^{n-2} b^2 + \ldots + \frac{n!}{0!n!} a^0 b^n
\]

**Example 4** Use a Factorial Form of the Binomial Theorem

Expand \((2x + y)^5\).

\[
(2x + y)^5 = \sum_{k=0}^{5} \frac{5!}{(5-k)!k!} (2x)^{5-k}y^k
\]

Binomial Theorem, factorial form

\[
= \frac{5!}{0!5!} (2x)^5 y^0 + \frac{5!}{4!1!} (2x)^4 y^1 + \frac{5!}{3!2!} (2x)^3 y^2 + \frac{5!}{2!3!} (2x)^2 y^3 + \frac{5!}{1!4!} (2x)^1 y^4 + \frac{5!}{0!5!} (2x)^0 y^5
\]

Let \( k = 0, 1, 2, 3, 4, \) and 5.

\[
= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} (2x)^5 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} (2x)^4 y + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1 \cdot 1} (2x)^3 y^2 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1} (2x)^2 y^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} (2x)y^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} y^5
\]

\[
= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5
\]

Simplify.

Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, \( k = 0 \) for the first term, \( k = 1 \) for the second term, and so on. In general, the value of \( k \) is always one less than the number of the term you are finding.
Find a Particular Term

Find the fifth term in the expansion of \((p + q)^{10}\).

First, use the Binomial Theorem to write the expansion in sigma notation.

\[
(p + q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10-k)k!} p^{10-k} q^k
\]

In the fifth term, \(k = 4\).

\[
\frac{10!}{(10-4)4!} p^{10-4} q^4 = \frac{10!}{6!4!} p^6 q^4 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} p^6 q^4
\]

\[
= 210 p^6 q^4
\]

Simplify.

Example 5

Find a Particular Term

Find the fifth term in the expansion of \((p + q)^{10}\).

First, use the Binomial Theorem to write the expansion in sigma notation.

\[
(p + q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10-k)k!} p^{10-k} q^k
\]

In the fifth term, \(k = 4\).

\[
\frac{10!}{(10-4)4!} p^{10-4} q^4 = \frac{10!}{6!4!} p^6 q^4 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} p^6 q^4
\]

\[
= 210 p^6 q^4
\]

Simplify.

Check for Understanding

Concept Check
1. List the coefficients in the row of Pascal’s triangle corresponding to \(n = 8\).
2. Identify the coefficient of \(a^{n-1}b\) in the expansion of \((a + b)^n\).
3. OPEN ENDED Write a power of a binomial for which the first term of the expansion is 625\(x^4\).

Guided Practice
Evaluate each expression.
4. 8!
5. \(\frac{13!}{9!}\)
6. \(\frac{12!}{2!10!}\)

Expand each power.
7. \((p + q)^5\)
8. \((t + 2)^6\)
9. \((x - 3y)^4\)

Find the indicated term of each expansion.
10. fourth term of \((a + b)^8\)
11. fifth term of \((2a + 3b)^{10}\)

Application
12. SCHOOL Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses?

Practice and Apply

Evaluate each expression.
13. 9!
14. 13!
15. \(\frac{9!}{7!}\)
16. \(\frac{7!}{4!}\)
17. \(\frac{12!}{8!4!}\)
18. \(\frac{14!}{5!9!}\)

Expand each power.
19. \((a - b)^3\)
20. \((m + n)^4\)
21. \((r + s)^6\)
22. \((m - a)^5\)
23. \((x + 3)^5\)
24. \((a - 2)^4\)
25. \((2b - x)^4\)
26. \((2a + b)^6\)
27. \((3x - 2y)^5\)
28. \((3x + 2y)^4\)
29. \(\left(\frac{a}{2} + 2\right)^5\)
30. \(\left(3 + \frac{m}{3}\right)^5\)

31. GEOMETRY Write an expanded expression for the volume of the cube at the right.

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Lesson 11-7 The Binomial Theorem 615
32. **GAMES** The diagram shows the board for a game in which ball bearings are dropped down a chute. A pattern of nails and dividers causes the bearings to take various paths to the sections at the bottom. For each section, how many paths through the board lead to that section?

33. **INTRAMURALS** Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two?

Find the indicated term of each expansion.

34. sixth term of \((x - y)^9\)  
35. seventh term of \((x + y)^{12}\)  
36. fourth term of \((x + 2)^7\)  
37. fifth term of \((a - 3)^8\)  
38. fifth term of \((2a + 3b)^{10}\)  
39. fourth term of \((2x + 3y)^9\)  
40. fourth term of \((x + \frac{1}{3})^7\)  
41. sixth term of \((x - \frac{1}{2})^{10}\)

42. **CRITICAL THINKING** Explain why \(\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}\) without finding the value of any of the expressions.

43. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How does a power of a binomial describe the numbers of boys and girls in a family?

Include the following in your answer:

- the expansion of \((b + g)^5\) and what it tells you about sequences of births of boys and girls in families with five children, and
- an explanation of how to find a formula for the number of sequences of births that have exactly \(k\) girls in a family of \(n\) children.

44. Which of the following represents the values of \(x\) that are solutions of the inequality \(x^2 < x + 20\)?

- A) \(x > -4\)
- B) \(x < 5\)
- C) \(-5 < x < 4\)
- D) \(-4 < x < 5\)

45. If four lines intersect as shown in the figure at the right, \(x + y = \)

- A) 70.
- B) 115.
- C) 140.
- D) It cannot be determined from the information given.
Find the sum of each geometric series. \((\text{Lessons 11-4 and 11-5})\)

1. \(a_1 = 5, r = 3, n = 12\)

2. \(\sum_{n=1}^{6} 2(-3)^{n-1}\)

3. \(\sum_{n=1}^{\infty} \frac{8}{3^n}\)

Find the first five terms of each sequence. \((\text{Lesson 11-6})\)

5. \(a_1 = 1, a_{n+1} = 2a_n + 3\)

6. \(a_1 = 2, a_{n+1} = a_n + 2n\)

7. Find the first three iterates of the function \(f(x) = -3x + 2\) for an initial value of \(x_0 = -1\). \((\text{Lesson 11-6})\)

Expand each power. \((\text{Lesson 11-7})\)

8. \((3x + y)^5\)

9. \((a + 2)^6\)

10. Find the fifth term of the expansion of \((2a + b)^9\). \((\text{Lesson 11-7})\)
**Proof and Mathematical Induction**

**What You’ll Learn**
- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

**Vocabulary**
- mathematical induction
- inductive hypothesis

**How does the concept of a ladder help you prove statements about numbers?**

Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?

**MATHEMATICAL INDUCTION** Mathematical induction is used to prove statements about positive integers. An induction proof consists of three steps.

**Key Concept**

**Mathematical Induction**

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Show that the statement is true for some integer $n$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 2</td>
<td>Assume that the statement is true for some positive integer $k$, where $k \geq n$. This assumption is called the <strong>inductive hypothesis</strong>.</td>
</tr>
<tr>
<td>Step 3</td>
<td>Show that the statement is true for the next integer $k + 1$.</td>
</tr>
</tbody>
</table>

**Example 1**

**Summation Formula**

Prove that the sum of the squares of the first $n$ positive integers is $\frac{n(n + 1)(2n + 1)}{6}$. That is, prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$.

**Step 1** When $n = 1$, the left side of the given equation is $1^2$ or 1. The right side is $\frac{1(1 + 1)(2(1) + 1)}{6}$ or 1. Thus, the equation is true for $n = 1$.

**Step 2** Assume $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k + 1)(2k + 1)}{6}$ for a positive integer $k$.

**Step 3** Show that the given equation is true for $n = k + 1$.

\[
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2 = \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2
\]

Add $(k + 1)^2$ to each side.

\[
= \frac{k(k + 1)(2k + 1) + 6(k + 1)^2}{6}
\]

Add.

\[
= \frac{(k + 1)[k(2k + 1) + 6(k + 1)]}{6}
\]

Factor.

\[
= \frac{(k + 1)[2k^2 + 7k + 6]}{6}
\]

Simplify.

\[
= \frac{(k + 1)(k + 2)(2k + 3)}{6}
\]

Factor.

\[
= \frac{(k + 1)(k + 1 + 1)(2(k + 1) + 1)}{6}
\]
The last expression on page 618 is the right side of the equation to be proved, where \( n \) has been replaced by \( k + 1 \). Thus, the equation is true for \( n = k + 1 \).

This proves that \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \) for all positive integers \( n \).

**Example 2 Divisibility**

Prove that \( 7^n - 1 \) is divisible by 6 for all positive integers \( n \).

**Step 1** When \( n = 1 \), \( 7^1 - 1 = 7^1 - 1 = 6 \). Since 6 is divisible by 6, the statement is true for \( n = 1 \).

**Step 2** Assume that \( 7^k - 1 \) is divisible by 6 for some positive integer \( k \). This means that there is a whole number \( r \) such that \( 7^k - 1 = 6r \).

**Step 3** Show that the statement is true for \( n = k + 1 \).

\[
\begin{align*}
7^k - 1 &= 6r & \text{Inductive hypothesis} \\
7^k &= 6r + 1 & \text{Add 1 to each side.} \\
7(7^k) &= 7(6r + 1) & \text{Multiply each side by 7.} \\
7^k + 1 &= 42r + 7 & \text{Simplify.} \\
7^k + 1 - 1 &= 42r + 6 & \text{Subtract 1 from each side.} \\
7^k + 1 - 1 &= 6(7r + 1) & \text{Factor.}
\end{align*}
\]

Since \( r \) is a whole number, \( 7r + 1 \) is a whole number. Therefore, \( 7^k + 1 - 1 \) is divisible by 6. Thus, the statement is true for \( n = k + 1 \).

This proves that \( 7^n - 1 \) is divisible by 6 for all positive integers \( n \).

**Counterexamples** Of course, not every formula that you can write is true. A formula that works for a few positive integers may not work for every positive integer. You can show that a formula is not true by finding a counterexample. This often involves trial and error.

**Example 3 Counterexample**

Find a counterexample for the formula \( 1^4 + 2^4 + 3^4 + \ldots + n^4 = 1 + (4n - 4)^2 \).

Check the first few positive integers.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Left Side of Formula</th>
<th>Right Side of Formula</th>
<th>Value of Right Side of Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1^4 or 1</td>
<td>1 + [4(1) - 4]^2 = 1 + 0^2 or 1</td>
<td>true</td>
</tr>
<tr>
<td>2</td>
<td>1^4 + 2^4 = 1 + 16 or 17</td>
<td>1 + [4(2) - 4]^2 = 1 + 4^2 or 17</td>
<td>true</td>
</tr>
<tr>
<td>3</td>
<td>1^4 + 2^4 + 3^4 = 1 + 16 + 81 or 98</td>
<td>1 + [4(3) - 4]^2 = 1 + 64 or 65</td>
<td>false</td>
</tr>
</tbody>
</table>

The value \( n = 3 \) is a counterexample for the formula.

**Check for Understanding**

1. **Concept Check** Describe some of the types of statements that can be proved by using mathematical induction.

2. **Explain** the difference between mathematical induction and a counterexample.

3. **Open Ended** Write an expression of the form \( b^n - 1 \) that is divisible by 2 for all positive integers \( n \).
Guided Practice

Prove that each statement is true for all positive integers.

4. \[1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}\]
5. \[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}\]
6. \[4^n - 1\] is divisible by 3.
7. \[5^n + 3\] is divisible by 4.

Find a counterexample for each statement.
8. \[1 + 2 + 3 + \ldots + n = n^2\]
9. \[2^n + 2n\] is divisible by 4.

Application

10. PARTIES Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after \(n\) guests have arrived, a total of \(\frac{n(n-1)}{2}\) handshakes have taken place.

Practice and Apply

Prove that each statement is true for all positive integers.

11. \[1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)\]
12. \[2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n + 1)}{2}\]
13. \[1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}\]
14. \[1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}\]
15. \[\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)\]
16. \[\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)\]
17. \[8^n - 1\] is divisible by 7.
18. \[9^n - 1\] is divisible by 8.
19. \[12^n + 10\] is divisible by 11.
20. \[13^n + 11\] is divisible by 12.

More About . . .

21. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top \(n\) rows is \(n^2 + 5n\).

22. GEOMETRIC SERIES Use mathematical induction to prove the formula
\[a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}\] for the sum of a finite geometric series.

23. ARITHMETIC SERIES Use mathematical induction to prove the formula
\[a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (n - 1)d] = \frac{n}{2}[2a_1 + (n - 1)d]\] for the sum of an arithmetic series.

24. PUZZLES Show that a \(2^n\) by \(2^n\) checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right.
Find a counterexample for each statement.

25. \(1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(3n - 1)}{2}\)

26. \(1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = 12n^3 - 23n^2 + 12n\)

27. \(3^n + 1\) is divisible by 4.

28. \(2^n + 2n^2\) is divisible by 4.

29. \(n^2 - n + 11\) is prime.

30. \(n^2 + n + 41\) is prime.

31. **CRITICAL THINKING** Refer to Example 2. Explain how to use the Binomial Theorem to show that \(7^n - 1\) is divisible by 6 for all positive integers \(n\).

32. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How does the concept of a ladder help you prove statements about numbers?

Include the following in your answer:

- an explanation of which part of an inductive proof corresponds to stepping onto the bottom step of the ladder, and
- an explanation of which part of an inductive proof corresponds to climbing from one step on the ladder to the next.

33. \[\frac{x - \frac{4}{x}}{1 - \frac{4}{x} + \frac{1}{x^2}} = \]

   \[\begin{array}{ll}
   \text{A} & \frac{x}{x-2} \\
   \text{B} & \frac{x^2 + 2}{x-2} \\
   \text{C} & \frac{x^2 + 2x}{x-2} \\
   \text{D} & \frac{x^2 + 2x}{(x-2)^2} \\
   \end{array}\]

34. **Quantitative Comparison**

   Compare the quantity in Column A and the quantity in Column B. Then determine whether:

   - \(\text{A}\) the quantity in Column A is greater,
   - \(\text{B}\) the quantity in Column B is greater,
   - \(\text{C}\) the two quantities are equal, or
   - \(\text{D}\) the relationship cannot be determined from the information given.

   \(PQRS\) is a square.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>length of QS</td>
</tr>
<tr>
<td></td>
<td>length of RS</td>
</tr>
</tbody>
</table>

**Standardized Test Practice**

35. \((x + y)^6\)  \(36. (a - b)^7\)  \(37. (2x + y)^8\)

Find the first three iterates of each function for the given initial value.

38. \(f(x) = 3x - 2, \ x_0 = 2\)  \(39. f(x) = 4x^2 - 2, \ x_0 = 1\)

40. **BIOLOGY** Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas?

   \(\text{(Lesson 10-2)}\)

41. \(\frac{1}{y + 1} - \frac{3}{y - 3} = 2\)  \(42. \frac{6}{a - 7} = \frac{a - 49}{a^2 - 7a} + \frac{1}{a}\)

**Maintain Your Skills**

**Mixed Review** (Lesson 11-7)

35. \((x + y)^6\)  \(36. (a - b)^7\)  \(37. (2x + y)^8\)

Find the first three iterates of each function for the given initial value.

(Lesson 11-6)

38. \(f(x) = 3x - 2, \ x_0 = 2\)  \(39. f(x) = 4x^2 - 2, \ x_0 = 1\)

40. **BIOLOGY** Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas?

   \(\text{(Lesson 10-2)}\)

Solve each equation. Check your solutions. (Lesson 9-6)

41. \(\frac{1}{y + 1} - \frac{3}{y - 3} = 2\)  \(42. \frac{6}{a - 7} = \frac{a - 49}{a^2 - 7a} + \frac{1}{a}\)

[www.algebra2.com/self_check_quiz](http://www.algebra2.com/self_check_quiz)
### Vocabulary and Concept Check

Choose the term from the list above that best completes each statement.

1. An(n) ____________ of an infinite series is the sum of a certain number of terms.
2. If a sequence has a common ratio, then it is a(n) _____________.
3. Using ___________, the series $2 + 5 + 8 + 11 + 14$ can be written as $\sum_{n=1}^{5} (3n - 1)$.
4. Eleven and 17 are the two ____ between 5 and 23 in the sequence 5, 11, 17, 23.
5. Using the ____________, $(a - 2)^4$ can be expanded to $a^4 - 8a^3 + 24a^2 - 32a + 16$.
6. The ___________ of the sequence $3, 2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}$ is $\frac{2}{3}$.
7. The ________ $11 + 16.5 + 22 + 27.5 + 33$ has a sum of 110.
8. A(n) ____ is expressed as $n! = n(n - 1)(n - 2) \ldots 2 \cdot 1$.

### Lesson-by-Lesson Review

#### 11-1 Arithmetic Sequences

**Concept Summary**

- An arithmetic sequence is formed by adding a constant to each term to get the next term.
- The $n$th term $a_n$ of an arithmetic sequence with first term $a_1$ and common difference $d$ is given by $a_n = a_1 + (n - 1)d$, where $n$ is any positive integer.

**Examples**

1. Find the 12th term of an arithmetic sequence if $a_1 = -17$ and $d = 4$.
   
   \[ a_n = a_1 + (n - 1)d \quad \text{Formula for the } n\text{th term} \]
   
   \[ a_{12} = -17 + (12 - 1)4 \quad n = 12, \quad a_1 = -17, \quad d = 4 \]
   
   \[ a_{12} = 27 \quad \text{Simplify.} \]

2. Find the two arithmetic means between 4 and 25.
   
   \[ a_n = a_1 + (n - 1)d \quad \text{Formula for the } n\text{th term} \]
   
   \[ a_4 = 4 + (4 - 1)d \quad n = 4, \quad a_1 = 4 \]
   
   \[ 25 = 4 + 3d \quad a_4 = 25 \]
   
   \[ 7 = d \quad \text{The arithmetic means are } 4 + 7 \text{ or } 11 \text{ and } 11 + 7 \text{ or } 18. \]
Exercises  Find the indicated term of each arithmetic sequence.  See Example 2 on p. 579.
9. \(a_1 = 6, \; d = 8, \; n = 5\)  
10. \(a_1 = -5, \; d = 7, \; n = 22\)  
11. \(a_1 = 5, \; d = -2, \; n = 9\)  
12. \(a_1 = -2, \; d = -3, \; n = 15\)

Find the arithmetic means in each sequence.  See Example 4 on page 580.
13. \(\frac{7}{4}, \; \frac{7}{4}, \; \frac{7}{4}, \; \frac{7}{4}, \; 9\)  
14. \(12, \; \frac{12}{2}, \; \frac{12}{2}, \; \frac{12}{2}, \; \frac{12}{2}, \; 4\)  
15. \(9, \; \frac{9}{3}, \; \frac{9}{3}, \; \frac{9}{3}, \; \frac{9}{3}, \; \frac{9}{3}, \; \frac{9}{3}, \; 6\)  
16. \(56, \; \frac{56}{2}, \; \frac{56}{2}, \; \frac{56}{2}, \; \frac{56}{2}, \; 28\)

11-2  
**Arithmetic Series**

**Concept Summary**
- The sum \(S_n\) of the first \(n\) terms of an arithmetic series is given by \(S_n = \frac{n}{2}[2a_1 + (n - 1)d]\) or \(S_n = \frac{n}{2}(a_1 + a_n)\).

**Example**

Find \(S_9\) for the arithmetic series with \(a_1 = 34\), \(a_n = 2\), and \(n = 9\).

\[
S_n = \frac{n}{2}(a_1 + a_n) \quad \text{Sum formula}
\]

\[
S_9 = \frac{9}{2}(34 + 2) = 9a_1 + 42 \quad n = 9, \; a_1 = 34, \; a_n = 2
\]

\[
S_9 = 162 \quad \text{Simplify.}
\]

**Exercises**  Find \(S_n\) for each arithmetic series.  See Examples on pages 584 and 585.
17. \(a_1 = 12, \; a_n = 117, \; n = 36\)  
18. \(4 + 10 + 16 + \ldots + 106\)  
19. \(10 + 4 + (-2) + \ldots + (-50)\)  
20. \(\sum_{n=2}^{13} (3n + 1)\)

11-3  
**Geometric Sequences**

**Concept Summary**
- A geometric sequence is one in which each term after the first is found by multiplying the previous term by a common ratio.
- The \(n\)th term \(a_n\) of a geometric sequence with first term \(a_1\) and common ratio \(r\) is given by \(a_n = a_1 \cdot r^{n-1}\), where \(n\) is any positive integer.

**Examples**

1 Find the fifth term of a geometric sequence for which \(a_1 = 7\) and \(r = 3\).

\[
a_n = a_1 \cdot r^{n-1} \quad \text{Formula for \(n\)th term}
\]

\[
a_5 = 7 \cdot 3^4 = 567 \quad n = 5, \; a_1 = 7, \; r = 3
\]

The fifth term is 567.

2 Find two geometric means between 1 and 8.

\[
a_n = a_1 \cdot r^{n-1} \quad \text{Formula for \(n\)th term}
\]

\[
a_4 = 1 \cdot r^3 \quad n = 4 \; \text{and} \; a_1 = 1
\]

\[
8 = r^3 \quad a_4 = 8
\]

The geometric means are 1(2) or 2 and 2(2) or 4.
Exercises  Find the indicated term of each geometric sequence. See Example 2 on page 589.
21. \( a_1 = 2, r = 2, n = 5 \)  
22. \( a_1 = 7, r = 2, n = 4 \)  
23. \( a_1 = 243, r = -\frac{1}{3}, n = 5 \)  
24. \( a_6 \) for \( \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots \)

Find the geometric means in each sequence. See Example 5 on page 590.
25. 3, _, _, _, 24
26. 7.5, _, _, _, _, 120
27. 8, _, _, _, _, 64
28. 5, _, _, _, _, _, 80

Geometric Series

Concept Summary
- The sum \( S_n \) of the first \( n \) terms of a geometric series is given by
  \[ S_n = \frac{a_1(1 - r^n)}{1 - r} \] or \( S_n = \frac{a_1 - a_1r^n}{1 - r} \), where \( r \neq 1 \).

Example
Find the sum of a geometric series for which \( a_1 = 7, r = 3, \) and \( n = 14 \).
\[ S_n = \frac{a_1 - a_1r^n}{1 - r} \quad \text{Sum formula} \]
\[ S_{14} = \frac{7 - 7 \cdot 3^{14}}{1 - 3} \quad n = 14, a_1 = 7, r = 3 \]
\[ S_{14} = 16,740,388 \quad \text{Use a calculator.} \]

Exercises  Find \( S_n \) for each geometric series. See Examples 1 and 3 on pages 595 and 596.
29. \( a_1 = 12, r = 3, n = 5 \)  
30. \( 4 - 2 + 1 - \ldots \) to 6 terms
31. \( 256 + 192 + 144 + \ldots \) to 7 terms
32. \( \sum_{n=1}^{5} \left(-\frac{1}{2}\right)^n - 1 \)

Infinite Geometric Series

Concept Summary
- The sum \( S \) of an infinite geometric series with \(-1 < r < 1\) is given by
  \[ S = \frac{a_1}{1 - r} \]

Example
Find the sum of the infinite geometric series for which \( a_1 = 18 \text{ and } r = -\frac{2}{7} \).
\[ S = \frac{a_1}{1 - r} \quad \text{Sum formula} \]
\[ = \frac{18}{1 - \left(-\frac{2}{7}\right)} = \frac{18}{\frac{9}{7}} = \frac{14}{7} \quad a_1 = 18, r = -\frac{2}{7} \]
\[ = \frac{14}{7} \quad \text{Simplify.} \]
Chapter 11 Study Guide and Review

Exercises Find the sum of each infinite geometric series, if it exists. See Example 1 on page 600.

33. \(a_1 = 6, r = \frac{11}{12}\)  
34. \(\frac{1}{8} - \frac{3}{16} + \frac{9}{32} - \frac{27}{64} + \ldots\)  
35. \(\sum_{n=1}^{\infty} -2\left(-\frac{5}{8}\right)^n - 1\)

11-6 Recursion and Special Sequences

Concept Summary
- In a recursive formula, each term is formulated from one or more previous terms.
- Iteration is the process of composing a function with itself repeatedly.

Examples

1 Find the first five terms of the sequence in which \(a_1 = 2\) and \(a_{n+1} = 2a_n - 1\).

\[
a_{n+1} = 2a_n - 1 \quad \text{Recursive formula}
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
<th>(a_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>(2(2) - 1) or 3</td>
<td>(2(3) - 1) or 5</td>
<td>(2(5) - 1) or 9</td>
<td>(2(9) - 1) or 17</td>
</tr>
</tbody>
</table>

The first five terms of the sequence are 2, 3, 5, 9, and 17.

2 Find the first three iterates of \(f(x) = -5x - 1\) for an initial value of \(x_0 = -1\).

\[
x_1 = f(x_0) \quad x_2 = f(x_1) \quad x_3 = f(x_2)
\]

\[
= f(-1) \quad = f(4) \quad = f(-21)
\]

\[
= -5(-1) - 1\ or\ 4 \quad = -5(4) - 1\ or\ -21 \quad = -5(-21) - 1\ or\ 104
\]

The first three iterates are 4, -21, and 104.

Exercises Find the first five terms of each sequence. See Example 1 on page 606.

36. \(a_1 = -2, a_{n+1} = a_n + 5\)  
37. \(a_1 = 3, a_{n+1} = 4a_n - 10\)  
38. \(a_1 = 2, a_{n+1} = a_n + 3n\)  
39. \(a_1 = 1, a_2 = 3, a_{n+2} = a_{n+1} + a_n\)

Find the first three iterates of each function for the given initial value. See Example 3 on page 608.

40. \(f(x) = -2x + 3, x_0 = 1\)  
41. \(f(x) = 7x - 4, x_0 = 2\)  
42. \(f(x) = x^2 - 6, x_0 = -1\)  
43. \(f(x) = -2x^2 - x + 5, x_0 = -2\)

11-7 The Binomial Theorem

Concept Summary
- Pascal’s triangle can be used to find the coefficients in a binomial expansion.
- The Binomial Theorem: 
  \[(a + b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k\]
Example

Expand \((a - 2b)^4\).

\[
(a - 2b)^4 = \sum_{k=0}^{4} \frac{4!}{(4-k)!k!} a^{4-k}(-2b)^k \quad \text{Binomial Theorem}
\]

\[
= \frac{4!}{4!} a^4(-2b)^0 + \frac{4!}{3!1!} a^3(-2b)^1 + \frac{4!}{2!2!} a^2(-2b)^2 + \frac{4!}{1!3!} a^1(-2b)^3 + \frac{4!}{0!4!} a^0(-2b)^4
\]

\[
= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4 \quad \text{Simplify.}
\]

Exercises

Expand each power. See Examples 1, 2, and 4 on pages 613 and 614.

44. \((x + y)^3\) 45. \((x - 2)^4\) 46. \((3r + s)^5\)

Find the indicated term of each expansion. See Example 5 on page 615.

47. fourth term of \((x + 2y)^6\) 48. second term of \((4x - 5)^{10}\)

11-8

Proof and Mathematical Induction

Concept Summary

- Mathematical induction is a method of proof used to prove statements about the positive integers.

Example

Prove \(1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)\) for all positive integers \(n\).

Step 1 When \(n = 1\), the left side of the given equation is 1. The right side is \(\frac{1}{4}(5^1 - 1)\) or 1. Thus, the equation is true for \(n = 1\).

Step 2 Assume that \(1 + 5 + 25 + \ldots + 5^{k-1} = \frac{1}{4}(5^k - 1)\) for some positive integer \(k\).

Step 3 Show that the given equation is true for \(n = k + 1\).

\[
1 + 5 + 25 + \ldots + 5^{k-1} + 5^{(k+1)-1} = \frac{1}{4}(5^k - 1) + 5^{(k+1)-1} \quad \text{Add} \ 5^{(k+1)-1} \ \text{to each side.}
\]

\[
= \frac{1}{4}(5^k - 1) + 5^k \quad \text{Simplify the exponent.}
\]

\[
= \frac{5^k - 1 + 4 \cdot 5^k}{4} \quad \text{Common denominator}
\]

\[
= \frac{5 \cdot 5^k - 1}{4} \quad \text{Distributive Property}
\]

\[
= \frac{1}{4}(5^{(k+1)} - 1) \quad 5 \cdot 5^k = 5^{k+1}
\]

The last expression above is the right side of the equation to be proved, where \(n\) has been replaced by \(k + 1\). Thus, the equation is true for \(n = k + 1\).

This proves that \(1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)\) for all positive integers \(n\).

Exercises

Prove that each statement is true for all positive integers. See Examples 1 and 2 on pages 618 and 619.

49. \(1 + 2 + 4 + \ldots + 2^n - 1 = 2^n - 1\) 50. \(6^n - 1\) is divisible by 5.
Vocabulary and Concepts

Choose the correct term to complete each sentence.

1. A sequence in which each term after the first is found by adding a constant to the previous term is called a(n) (arithmetic, geometric) sequence.

2. A (Fibonacci sequence, series) is a sum of terms of a sequence.

3. (Recursive formulas, Binomial Theorem) and the Binomial Theorem can be used to expand powers of binomials.

Skills and Applications

4. Find the next four terms of the arithmetic sequence 42, 37, 32, ….

5. Find the 27th term of an arithmetic sequence for which \(a_1 = 2\) and \(d = 6\).

6. Find the three arithmetic means between \(4\) and \(16\).

7. Find the sum of the arithmetic series for which \(a_1 = 7\), \(n = 31\), and \(a_n = 127\).

8. Find the next two terms of the geometric sequence \(\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots\)

9. Find the sixth term of the geometric sequence for which \(a_1 = 5\) and \(r = -2\).

10. Find the two geometric means between 7 and 189.

11. Find the sum of the geometric series for which \(a_1 = 125\), \(r = \frac{2}{5}\), and \(n = 4\).

12. Find the sum of each series, if it exists.

13. \[\sum_{k=3}^{15} (14 - 2k)\]

14. \[\sum_{n=1}^{\infty} \frac{1}{3}(-2)^n - 1\]

15. \[12 + (-6) + 3 - \frac{3}{2} + \ldots\]

16. \(a_1 = 1, a_{n+1} = a_n + 3\)

17. \(a_1 = -3, a_{n+1} = a_n + n^2\)

18. Find the first three iterates of \(f(x) = x^2 - 3x\) for an initial value of \(x_0 = 1\).

19. Expand \((2s - 3t)^5\).

20. Find the third term of the expansion of \((x + y)^{10}\).

Prove that each statement is true for all positive integers.

21. \(1 + 3 + 5 + \ldots + (2n - 1) = n^2\)

22. \(14^n - 1\) is divisible by 13.

23. DESIGN A landscaper is designing a wall of white brick and red brick. The pattern starts with 20 red bricks on the bottom row. Each row above it contains 3 fewer red bricks than the preceding row. If the top row contains no red bricks, how many rows are there and how many red bricks were used?

24. RECREATION One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will the balloon rise in 5 minutes?

25. STANDARDIZED TEST PRACTICE Find the next term in the geometric sequence 8, 6, \(\frac{9}{2}, \frac{27}{8}, \ldots\).

A) \(\frac{11}{8}\)

B) \(\frac{27}{16}\)

C) \(\frac{9}{4}\)

D) \(\frac{81}{32}\)
Part 1  Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. For all positive integers, let \( n = n + g \), where \( g \) is the greatest factor of \( n \), and \( g < n \). If \( 18 = x \), then \( x = \)


2. If \( p \) is positive, what percent of \( 6p \) is 12?

A \( \frac{p}{100} \)  B \( \frac{p}{2} \)  C \( \frac{12}{p} \)  D \( \frac{200}{p} \)

3. A box is 12 units tall, 6 units long, and 8 units wide. A designer is creating a new box that must have the same volume as the first box. If the length and width of the new box are each 50% greater than the length and width of the first box, about how many units tall will the new box be?

A 5.3  B 6.8  C 7.1  D 8.5

4. Which of the following statements must be true when \( 0 < m < 1 \)?

I \( \sqrt{m} > m \)  II \( 4m < 1 \)  III \( m^2 - m^3 < 0 \)

A I only  B III only  C I and II only  D I, II, and III

5. If \( 3kx - \frac{4s}{t} = 3ky \), then \( x - y = \)

A \( -\frac{4s}{3kt} \)  B \( -\frac{4s}{t} + \frac{1}{3k} \)  C \( \frac{4s}{3lt} - k \)  D \( \frac{4s}{3kt} \)

6. For all \( n \neq 0 \), what is the slope of the line passing through \( (3n, -k) \) and \( (-n, -k) \)?

A 0  B \( \frac{k}{2n} \)  C \( \frac{2n}{k} \)  D undefined

7. Which is the graph of the equation \( x^2 + (y - 4)^2 = 20 \)?

A line  B parabola  C circle  D ellipse

8. \( \frac{x - \frac{9}{x}}{1 - \frac{6}{x} + \frac{9}{x^2}} = \)

A \( \frac{x}{x - 3} \)  B \( \frac{x^2 + 3}{x - 3} \)  C \( \frac{x^2 + 3x}{x - 3} \)  D \( \frac{x^2 + 3x}{(x - 3)^2} \)

9. What is the sum of the positive even factors of 30?

A 18  B 30  C 48  D 72

10. If \( \ell_1 \) is parallel to \( \ell_2 \) in the figure, what is the value of \( x \)?

A 30  B 40  C 70  D 80

Test-Taking Tip

Question 5  Some questions ask you to find the value of an expression. It is often not necessary to find the value of each variable in the expression. For example, to answer Question 5, it is not necessary to find the values of \( x \) and \( y \). Isolate the expression \( x - y \) on one side of the equation.
Part 2  Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. \[
\begin{align*}
\frac{AA}{CC} + \frac{BB}{CC}
\end{align*}
\]
If \(A, B,\) and \(C\) are each digits and \(A = 3B,\) then what is one possible value of \(C?\)

12. In the figure, each arc is a semicircle. If \(B\) is the midpoint of \(AD\) and \(C\) is the midpoint of \(BD,\) what is the ratio of the area of the semicircle \(CD\) to the area of the semicircle \(AD?\)

13. Two people are 17.5 miles apart. They begin to walk toward each other along a straight line at the same time. One walks at the rate of 4 miles per hour, and the other walks at the rate of 3 miles per hour. In how many hours will they meet?

14. If \(\frac{x + y}{x} = \frac{5}{4},\) then \(\frac{y}{x} = \)

15. A car’s gasoline tank is \(\frac{1}{2}\) full. After adding 7 gallons of gas, the gauge shows that the tank is \(\frac{3}{4}\) full. How many gallons does the tank hold?

16. If \(a = 15 - b,\) what is the value of \(3a + 3b?\)

17. If \(x^9 = \frac{45}{y}\) and \(x^7 = \frac{1}{5y},\) and \(x > 0,\) what is the value of \(x?\)

Part 3  Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- the quantity in Column A is greater,
- the quantity in Column B is greater,
- the two quantities are equal, or
- the relationship cannot be determined from the information given.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.</td>
<td>19.</td>
</tr>
<tr>
<td>three times the perimeter of Square A</td>
<td>the perimeter of Square B</td>
</tr>
</tbody>
</table>

19. The area of Square B is equal to nine times the area of Square A.

20. \(n\) = \(n(n + 1)\) if \(n\) is even

\(n\) = \(n(n - 1)\) if \(n\) is odd

21. \([1 - \sqrt{3}][1 - \sqrt{3}]\) \([1 - \sqrt{3}][1 + \sqrt{3}]\)

22. \[
\begin{align*}
\frac{x + y}{2} & \quad \frac{x + z}{2}
\end{align*}
\]